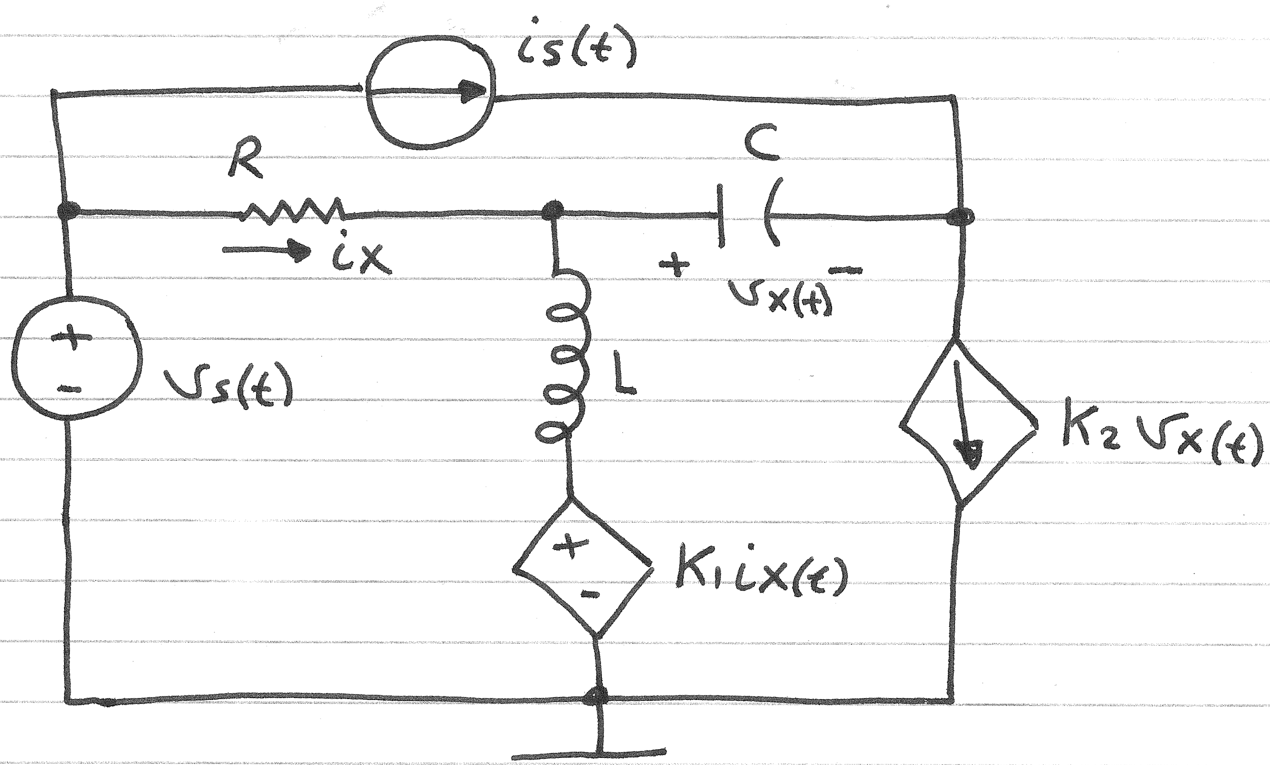
Network Analysis 1 ENEE 231

Ch2: Circuit Elements



Electric Circuit

Network : The interconnection of two or more simple circuit element is called electrical network

Circuit : If the network contains at least one closed path, it is called electric circuit

Circuit analysis : given a Circuit in which all the Components are specified, analysis involves finding such things as the voltage across some elements or the current through another.

The Solution is unique.

Circuit design involves determining the Circuit configuration that will meet certain specifications.

The Solution is not unique.

Circuit Elements

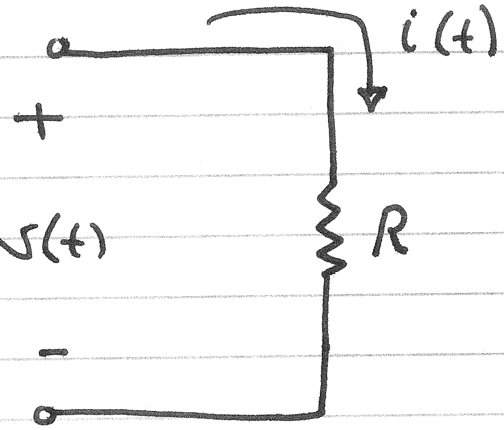
1) Active element : Capable of delivering power to some external elements.
(Sources)

2) Passive element : Capable only of recieving power.
(R, L, C, ...)

Circuit elements can be classified according to the realationship of the current through the element to the voltage a cross the element

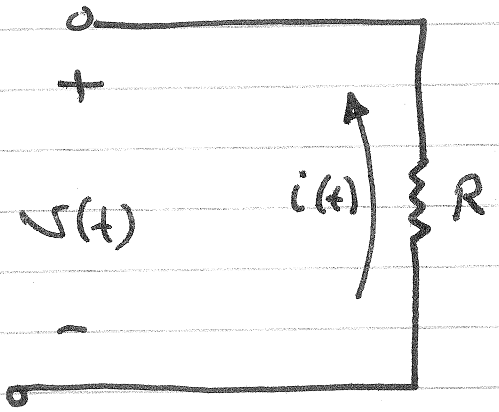
Circuit Elements

1) Resistor



$$v(t) = R\,i(t)$$

$$i(t) = \frac{1}{R}v(t) = G\,v(t)$$

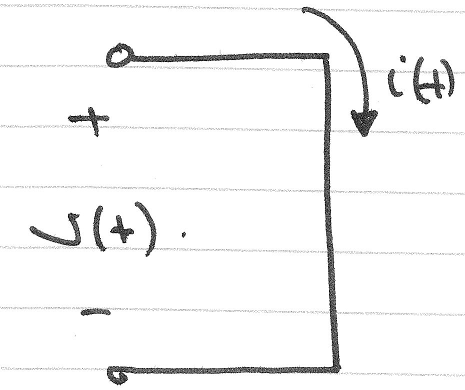


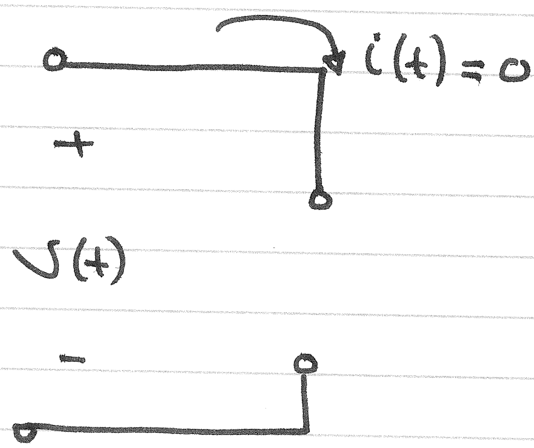
$$v(t) = -R\,i(t)$$

R is called the resistance of the component and is measured in units of ohm (Ω)

G is called the Conductance of the Component and is measured in units of Siemens ($\mho$)

* Two special resistor values





$v(t) = 0$	$i(t) = 0$
Short Circuit	open Circuit
$R = 0\ \Omega$	$R = \infty\ \Omega$
$G = \infty\ \mho$	$G = 0\ \mho$

Resistors and electric power

Resistors are passive elements that can only absorb energy.

$$p(t) = v(t)\, i(t)$$

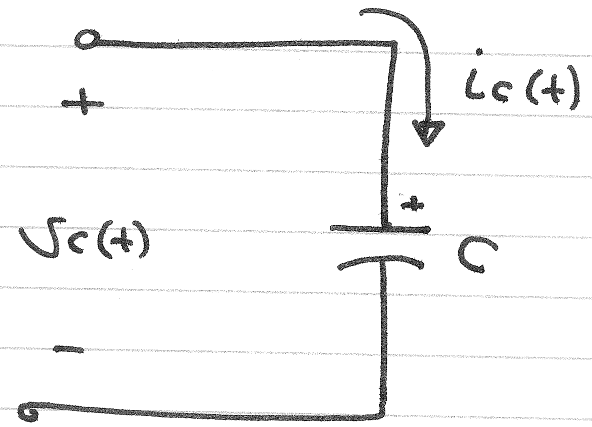
$$v(t) = R\, i(t)$$

$$\therefore \quad p(t) = \frac{v^2(t)}{R}$$

$$p(t) = R\, i^2(t)$$

Circuit Elements

2) Capacitors



$$v_c(t) = \frac{1}{C}\int_{-\infty}^{t} i_c(t)\,dt$$

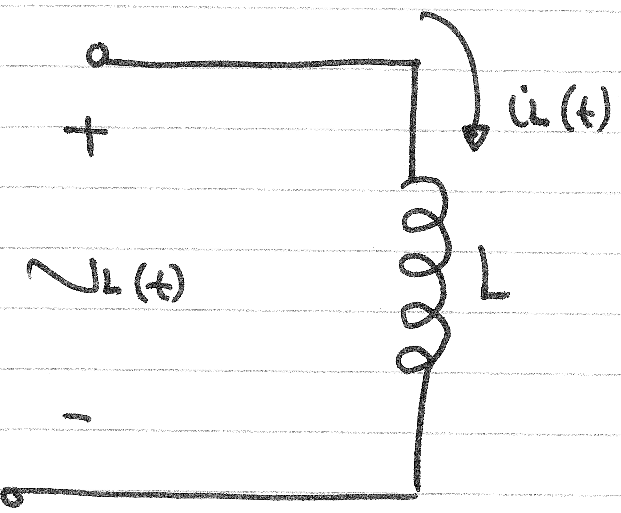
$$v_c(t) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i_c(t)\,dt \quad \text{for } t \geq 0$$

$$i_c(t) = C\,\frac{dv_c(t)}{dt}$$

C is called the Capacitance of the Capacitor and is measured in units of Farad (F)

Circuit Elements

3) Inductors



$$v_L(t) = L\,\frac{di_L(t)}{dt}$$

$$i_L(t) = i_L(0) + \frac{1}{L}\int_{0^-}^{t} v_L(t)\,dt \quad \text{for } t \geq 0$$

L is Called the inductance of the Coil and is measured in units of Henry (H)

Circuit Elements

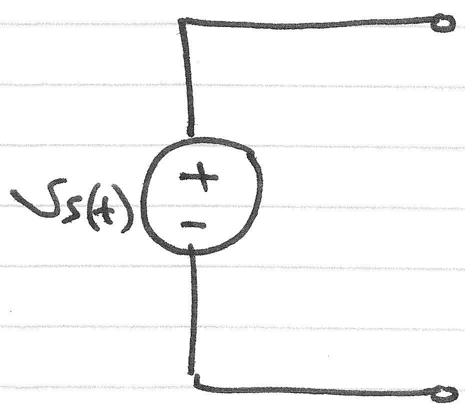
Active elements

- Independent Sources
- Dependent Sources

Independent Sources

1. Independent Voltage Source :

a Circuit element in which the voltage a cross its terminal is Completely independent of the Current through it.

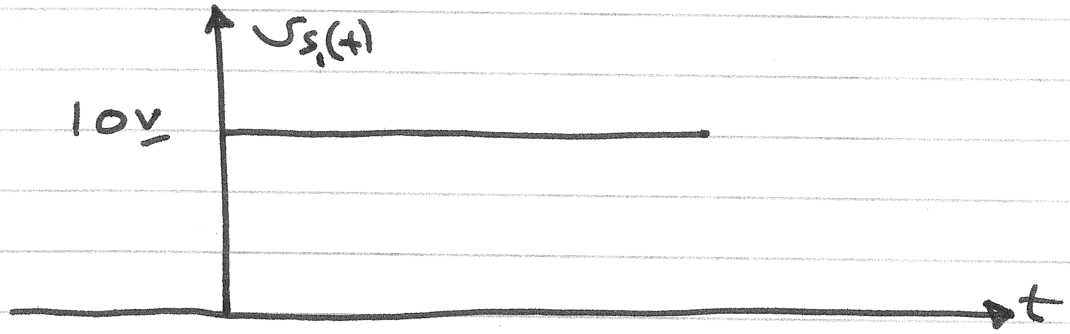


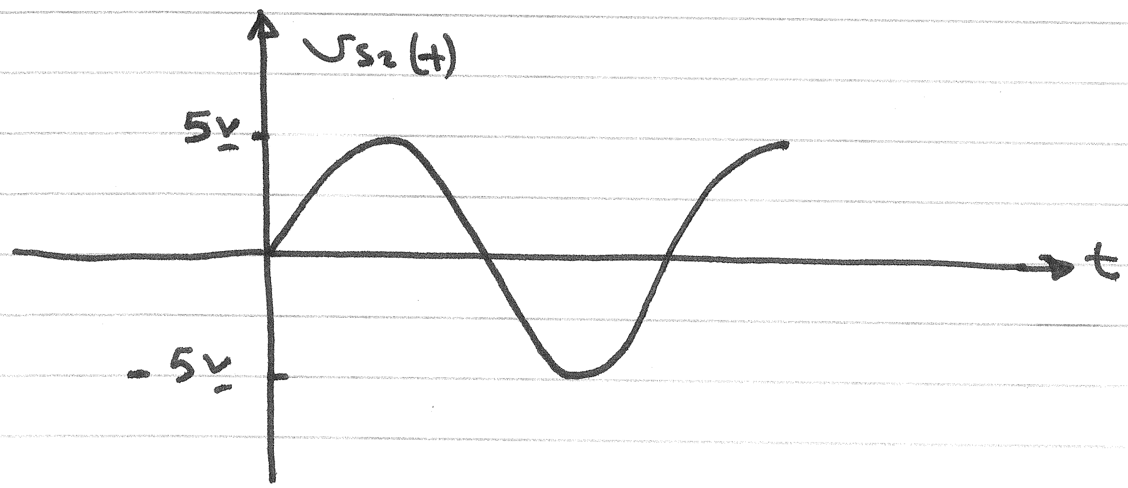
$V_s(t) = 10\,v$ (DC)

$V_s(t) = 5 \sin\omega t\ v$ (ac)

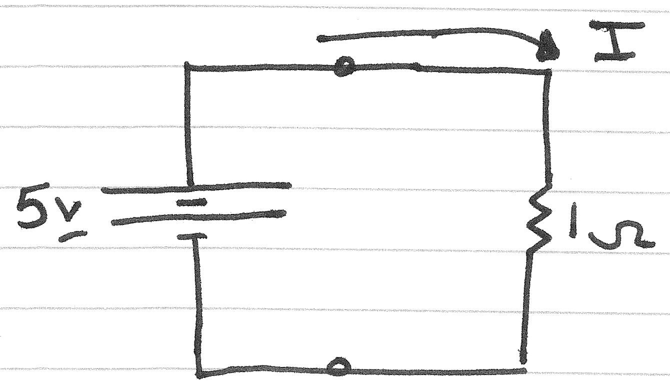
$V_s(t) = 10 e^{-t}\ v$

$v_{s_1}(t) = 10 \text{ V} \quad (DC)$

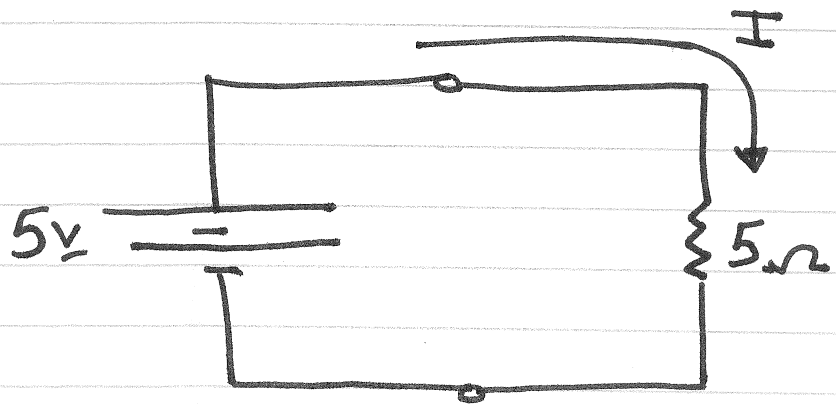




$v_{s_2}(t) = 5 \sin \omega t \text{ V} \quad (ac)$



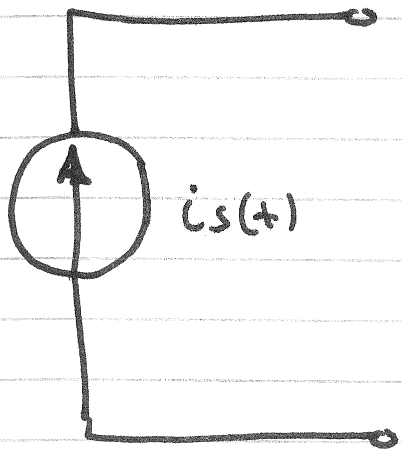
$$I = \frac{5v}{1\Omega} = 5A$$



$$I = \frac{5v}{5\Omega} = 1A$$

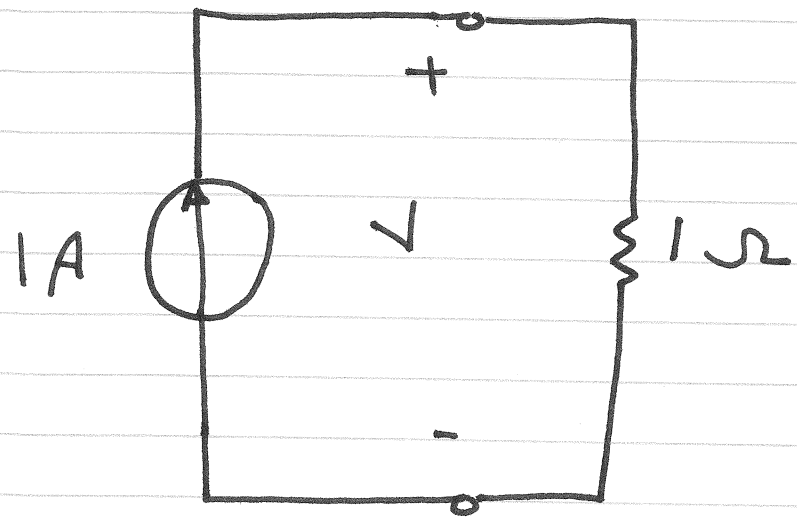
2. Independent Current Source :

a Circuit element in which the current through it is completely independent of the voltage across its terminals.

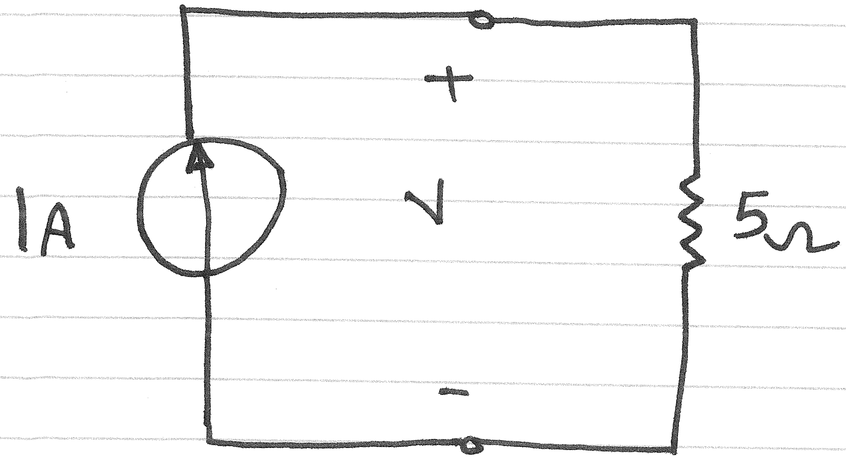


$i_s(t) = 10 \sin\omega t \ A$

$i_s(t) = 20 \ A$



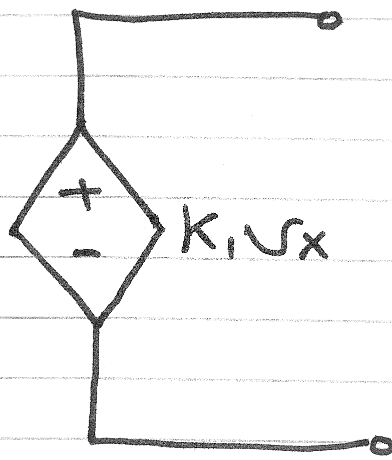
$$V = (1A)(1\Omega) = 1\underline{V}$$

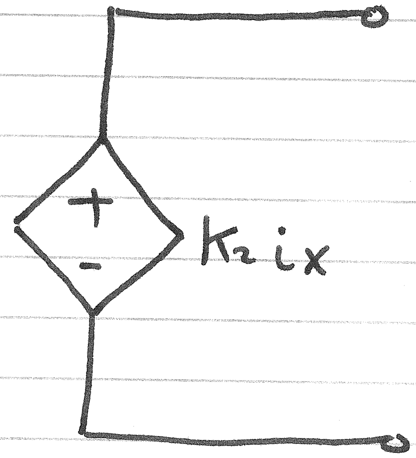


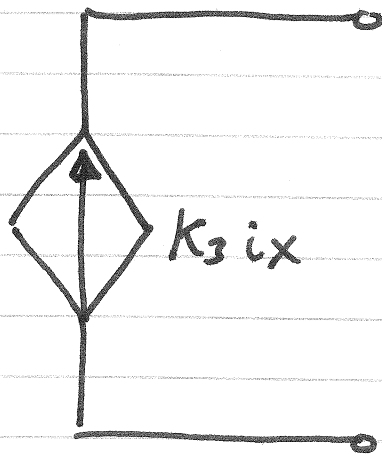
$$V = (1A)(5\Omega) = 5\underline{V}$$

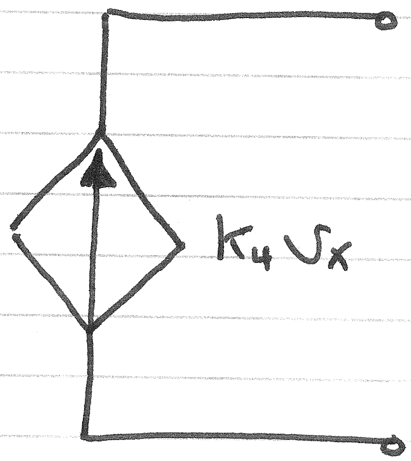
Dependent Sources :

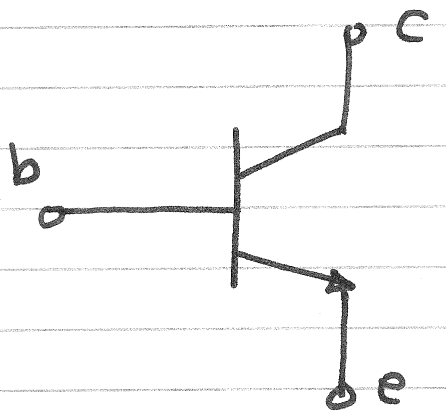
are Sources in which the Source voltage (or current) depend upon a current or voltage else where in the Circuit.



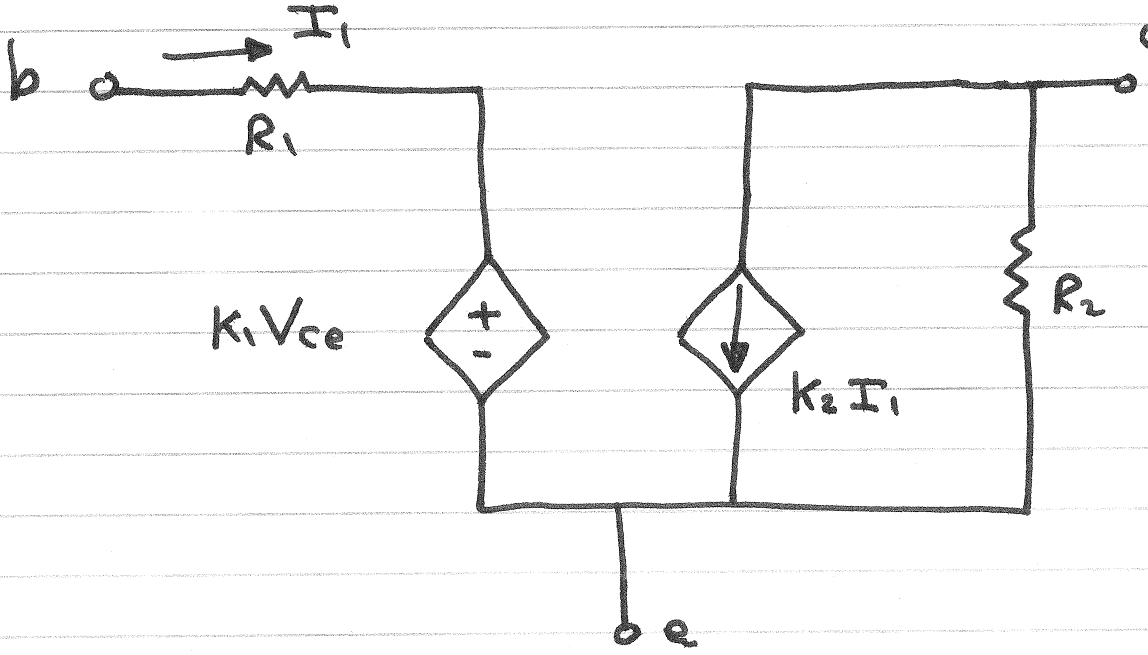








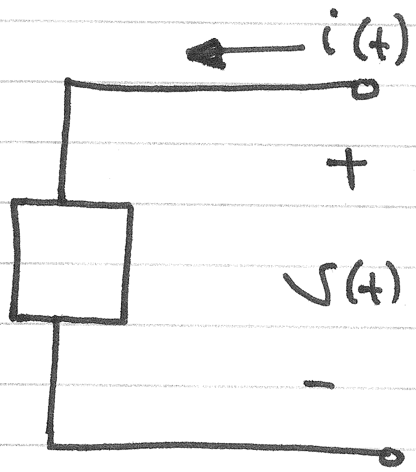
c
b
e



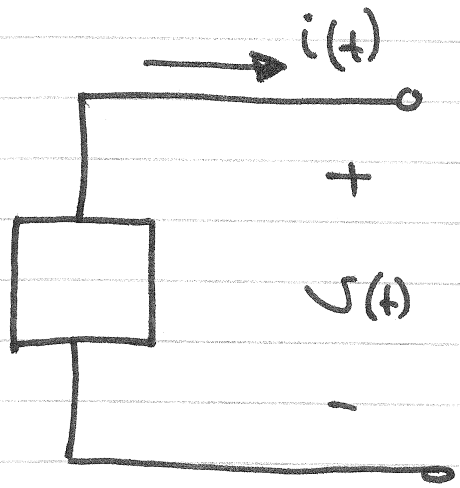
I_1
b
R_1
c
$K_1 V_{ce}$
R_2
$K_2 I_1$
e

Power and Energy

$$p(t) = \frac{dw(t)}{dt}$$



$$p(t) = +\,v(t)\,i(t)$$ absorbing



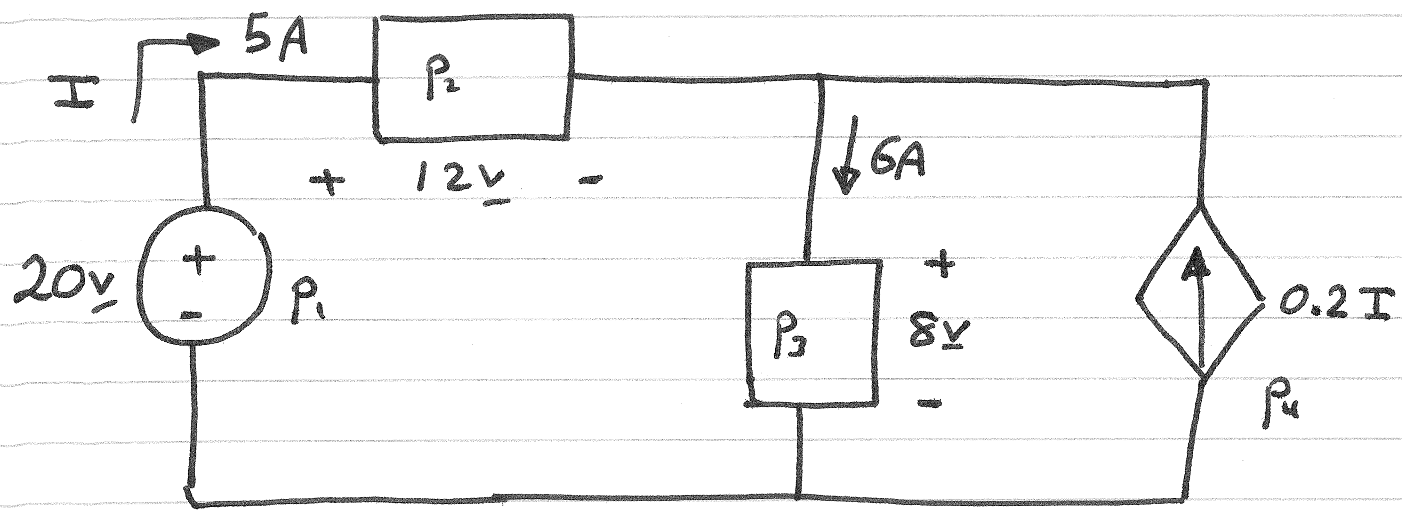
$$p(t) = -\,v(t)\,i(t)$$ Supplying

The Law of Conservation of energy must be obeyed in any electric Circuit.

The algebraic sum of power in a Circuit at any instant of time, must be Zero.

$$\sum p(t) = 0$$

Calculate the power supplied or absorbed by each element



$P_1 = (20)(-5) = -100\text{ W}$ supplied power

$P_2 = (12)(5) = 60\text{ W}$ absorbed power

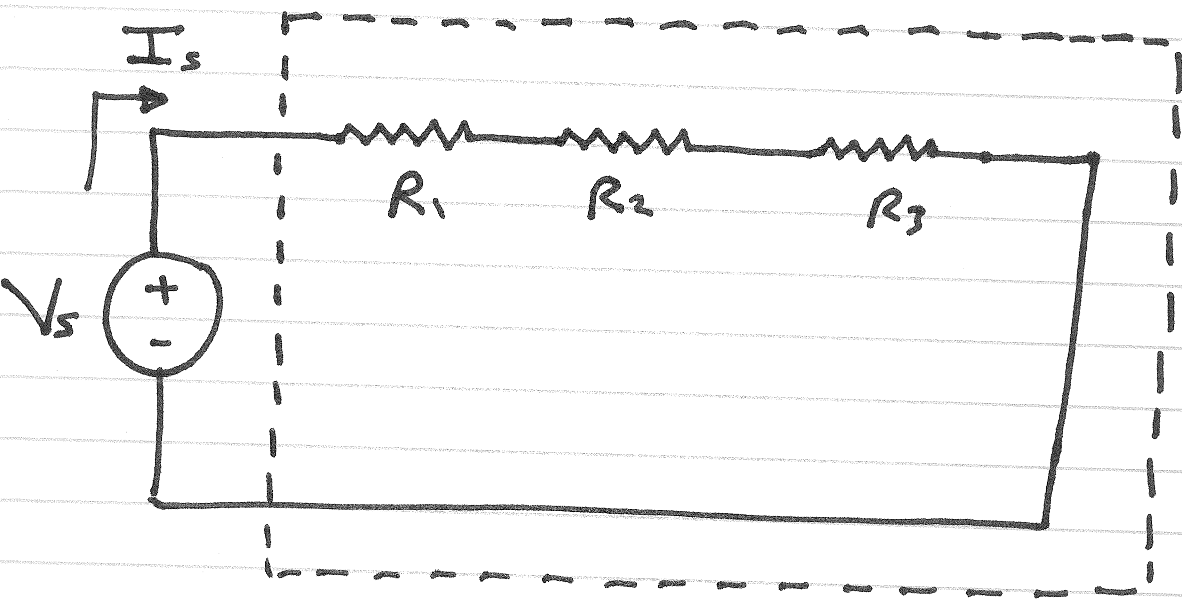
$P_3 = (8)(+6) = 48\text{ W}$ absorbed power

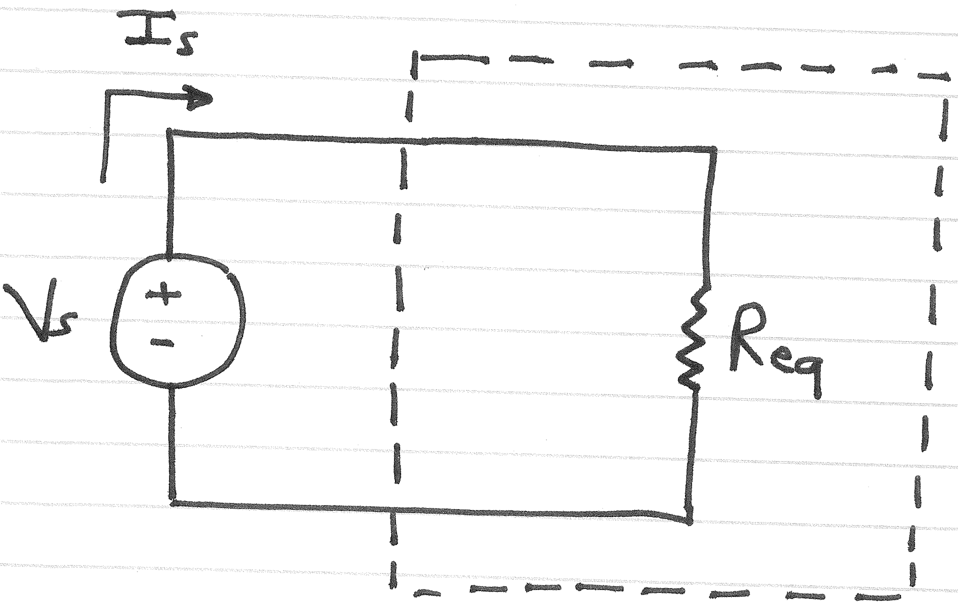
$P_4 = (8)(-0.2 \times 5) = -8\text{ W}$ Supplied

$$P_{absorbed} = P_{supplied}$$

$$60 + 48 = 100 + 8$$

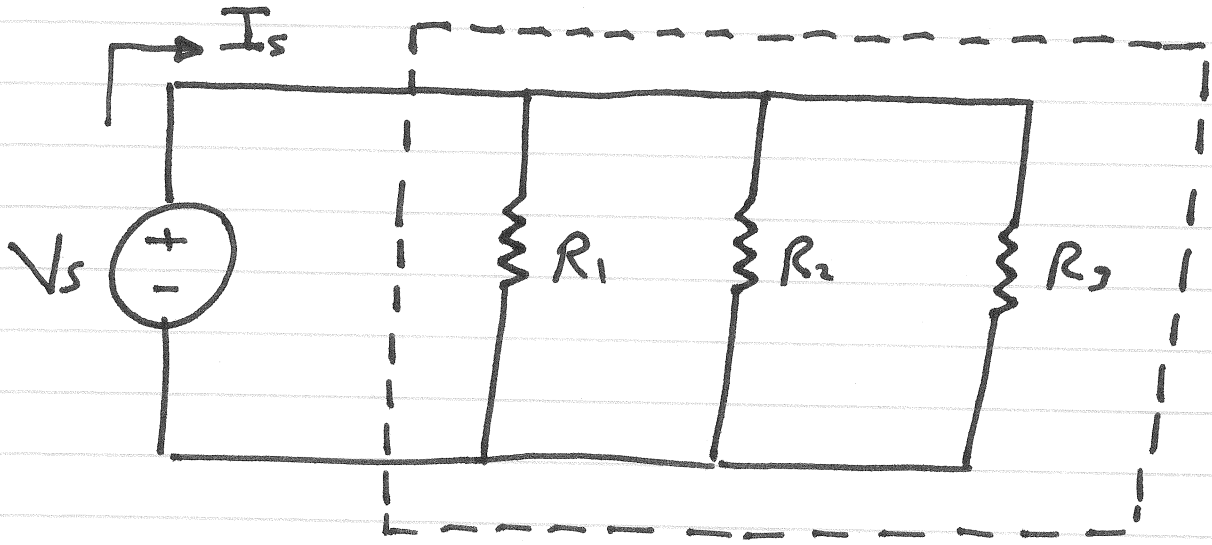
Resistors in Series

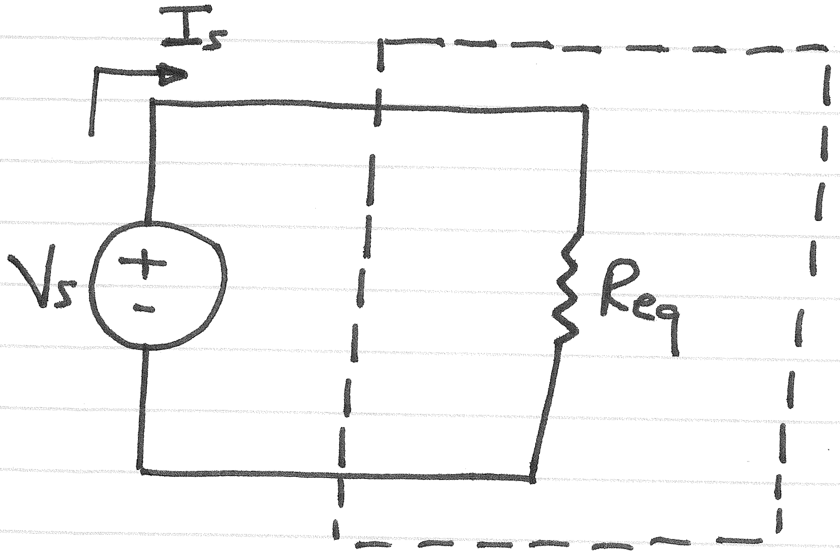




$$R_{eq} = R_1 + R_2 + R_3$$

Resistors in Parallel





$$\frac{1}{R_{eq}} = \frac{1}{R_1} + \frac{1}{R_2} + \frac{1}{R_3}$$

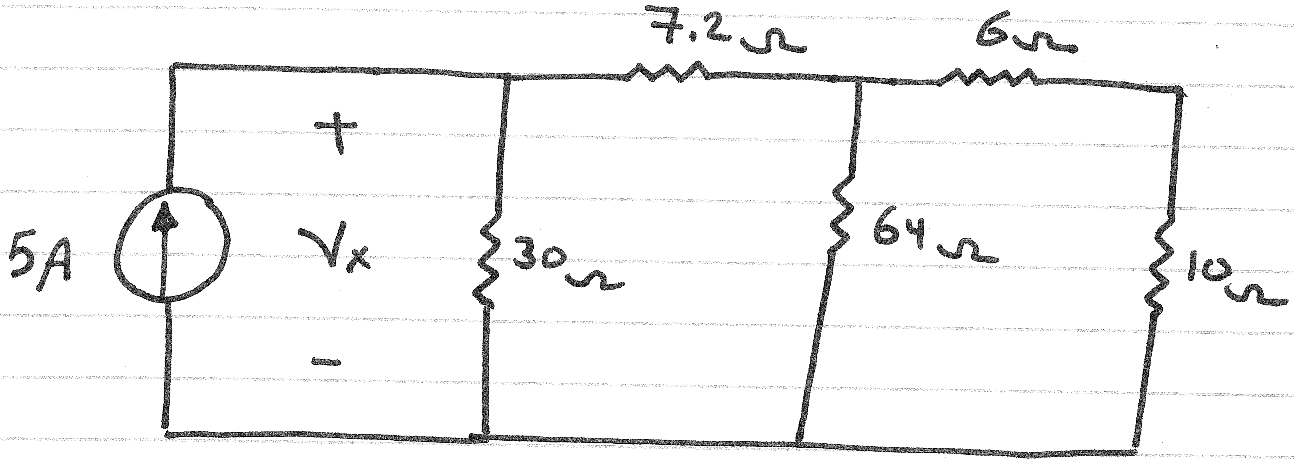
Two Resistors in Parallel

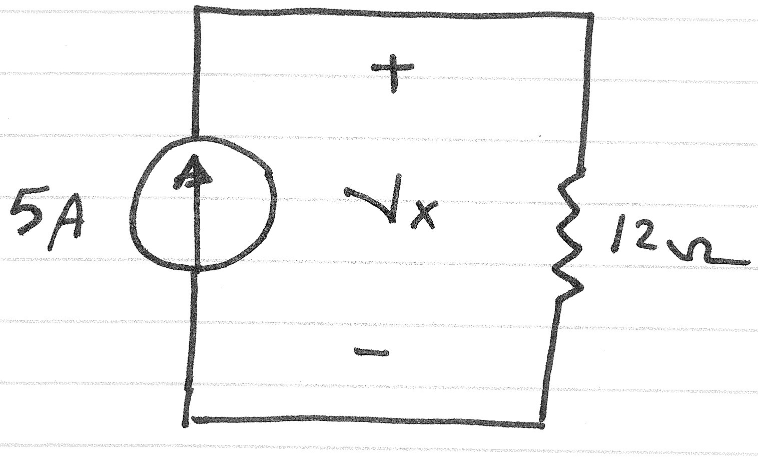
$$R_{eq} = R_1 \| R_2$$

$$R_{eq} = \frac{R_1 R_2}{R_1 + R_2}$$

$$0.5 \min(R_1, R_2) < R_1 \| R_2 < \min(R_1, R_2)$$

Find V_x



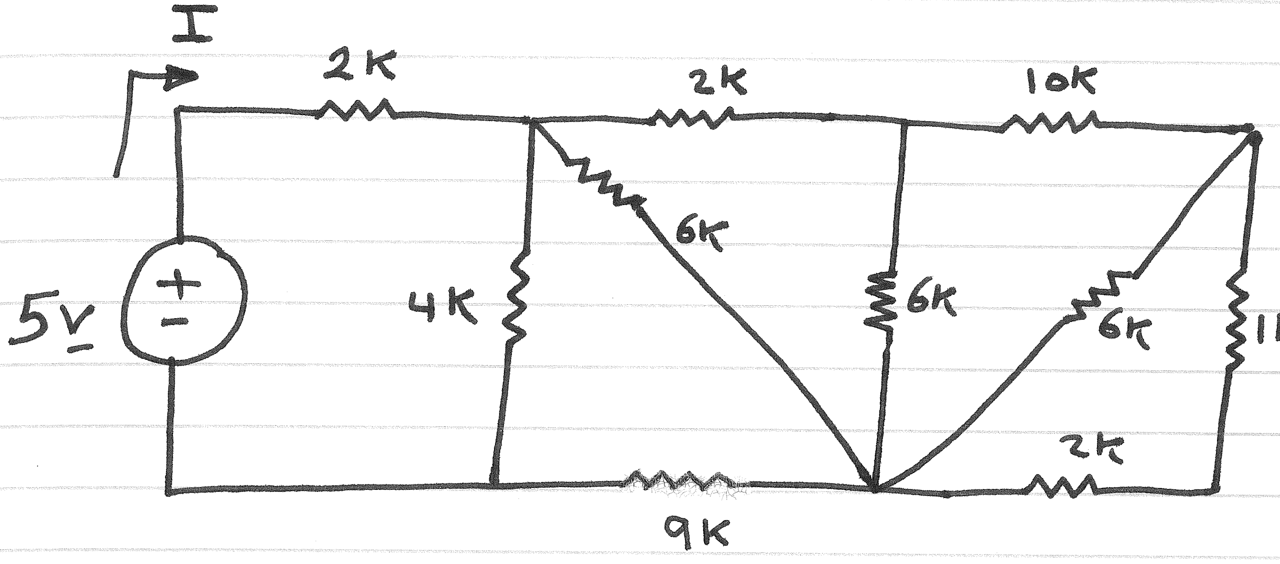


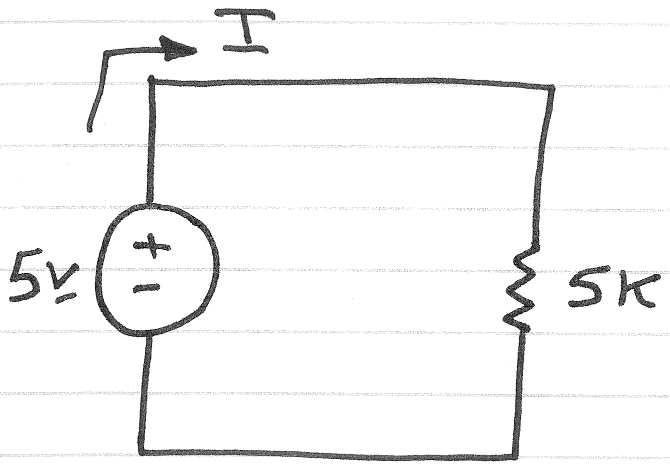
$$V_x = (5)(12) = 60\,\text{V}$$

$16\,\Omega \parallel 64 = 12.8\,\Omega$

$20\,\Omega \parallel 30\,\Omega = 12\,\Omega$

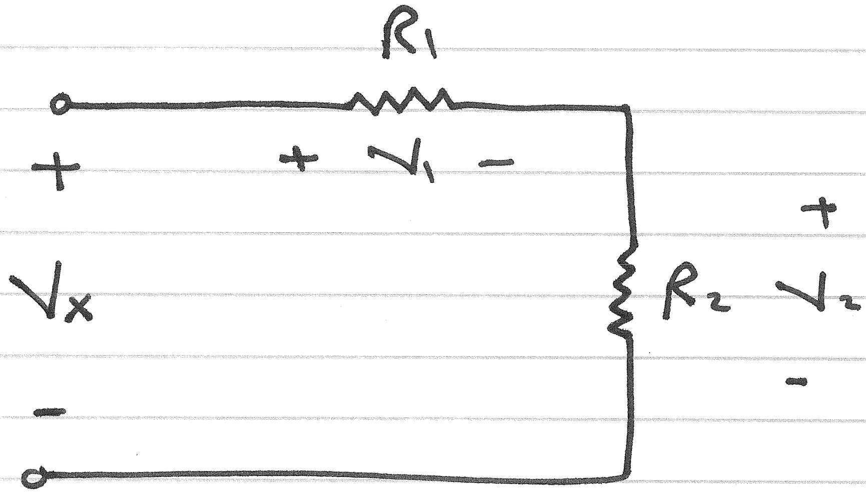
Find I





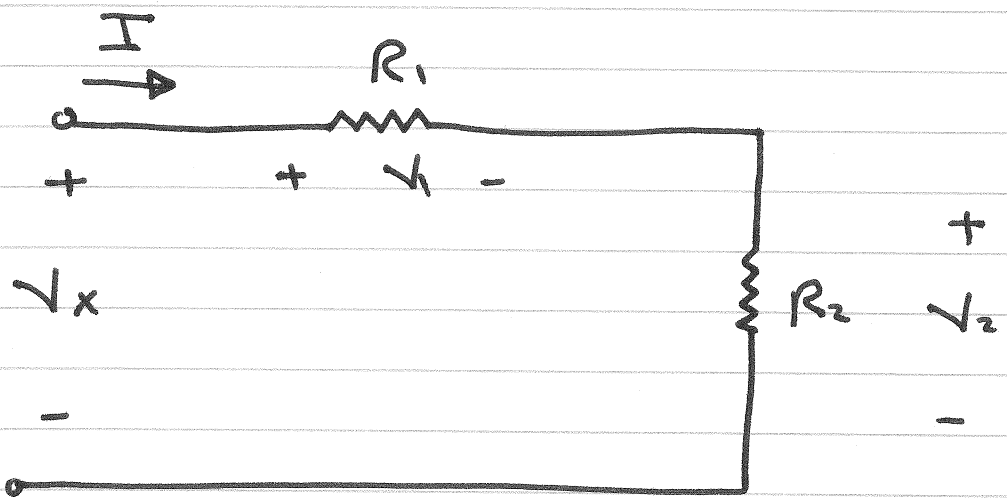
$$I = \frac{5v}{5K} = 1mA$$

Voltage Divider Rule



$$V_1 = \frac{R_1}{R_1 + R_2} V_x$$

$$V_2 = \frac{R_2}{R_1 + R_2} V_x$$



KVL :

$$V_x = V_1 + V_2$$

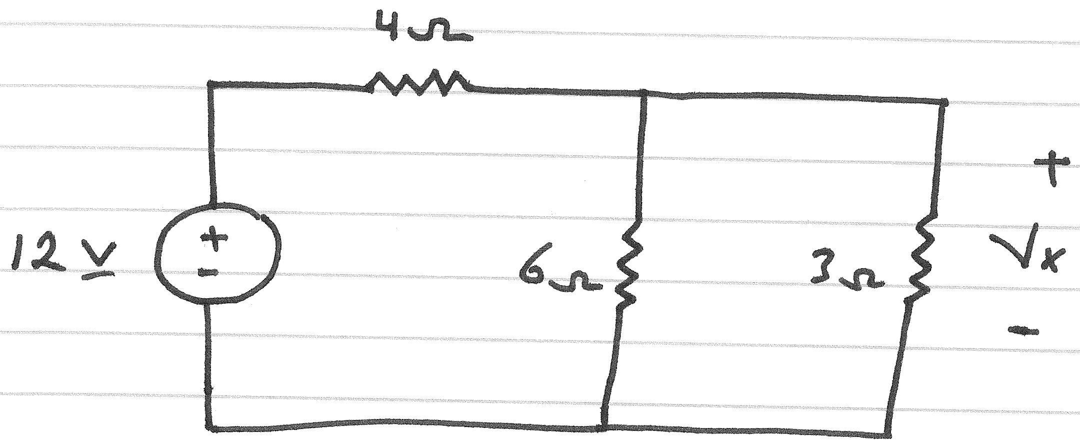
$$V_x = R_1 I + R_2 I$$

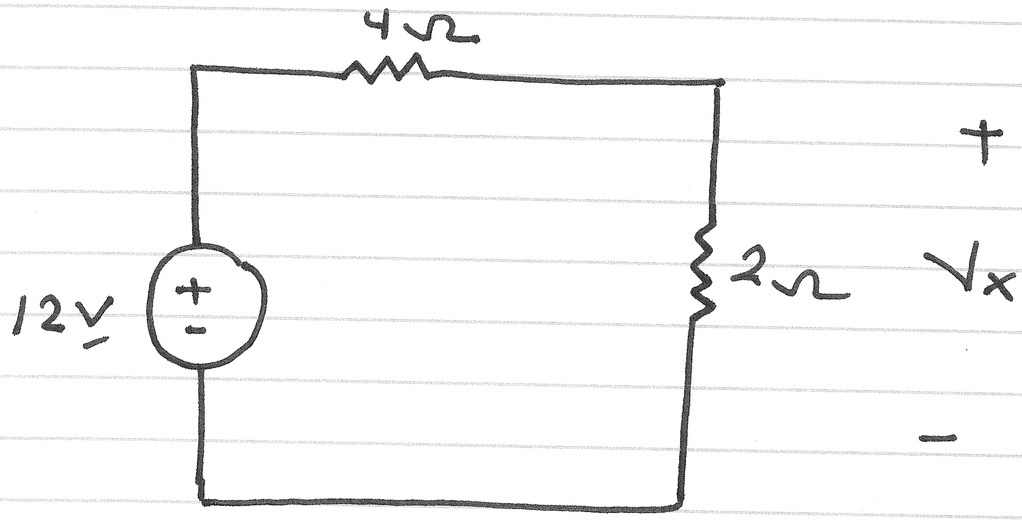
$$\therefore \quad I = \frac{V_x}{R_1 + R_2}$$

$$V_1 = R_1 I = \frac{R_1}{R_1 + R_2} V_x$$

$$V_2 = R_2 I = \frac{R_2}{R_1 + R_2} V_x$$

Find V_x

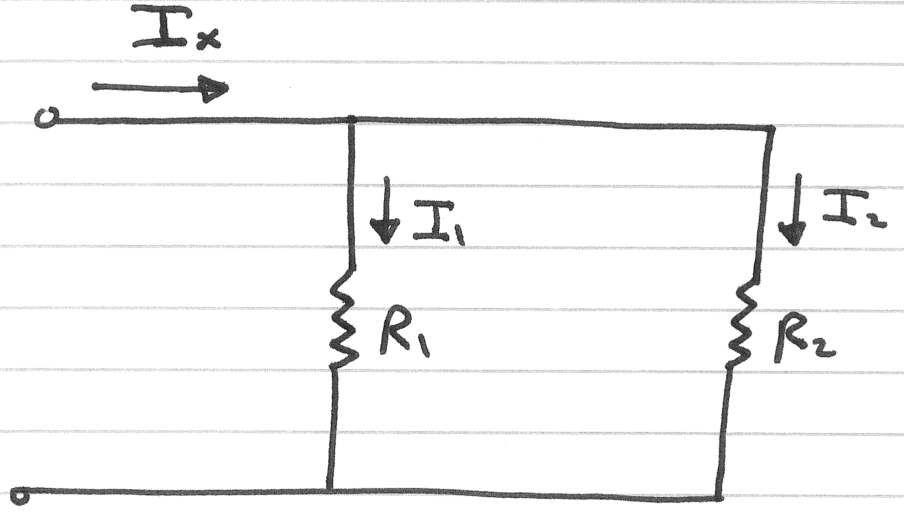




$$V_x = \frac{2}{4+2} \quad 12$$

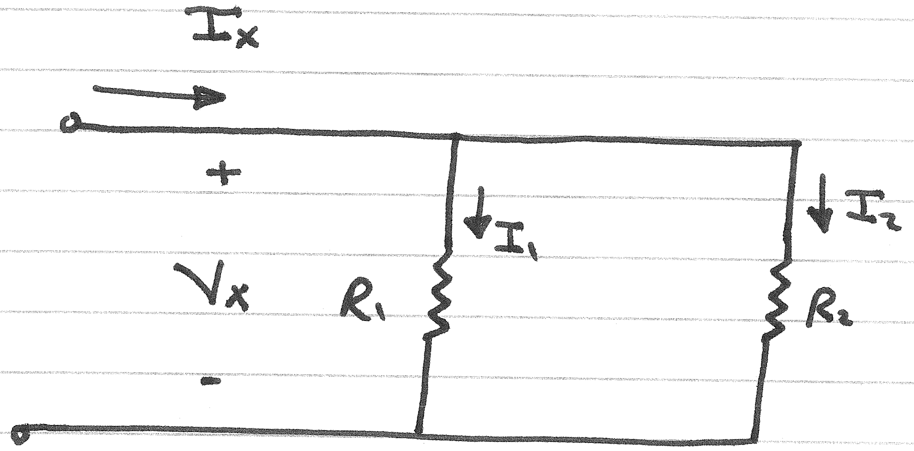
$$V_x = 4 \ \underline{V}$$

Current Divider Rule



$$I_1 = \frac{R_2}{R_1 + R_2} I_x$$

$$I_2 = \frac{R_1}{R_1 + R_2} I_x$$



KCL :

$$I_x = I_1 + I_2$$

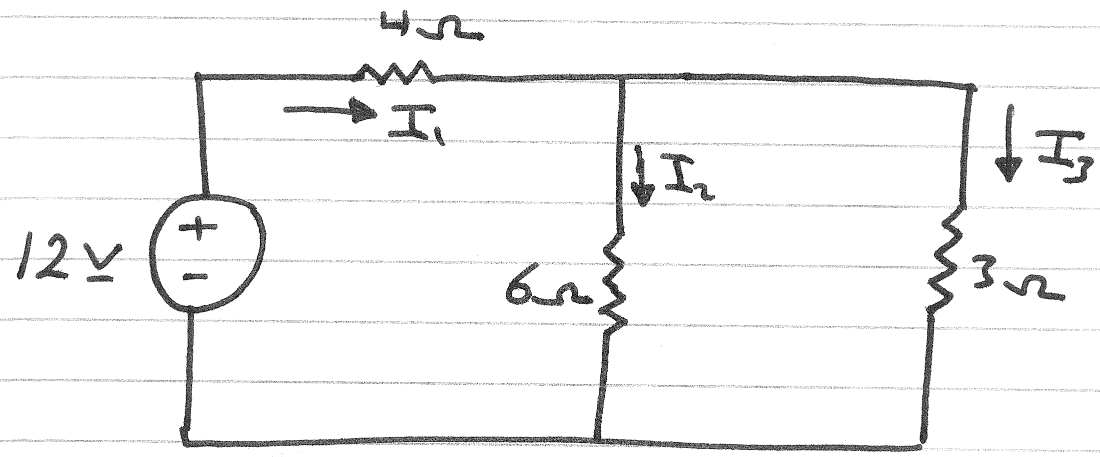
$$I_x = \frac{V_x}{R_1} + \frac{V_x}{R_2}$$

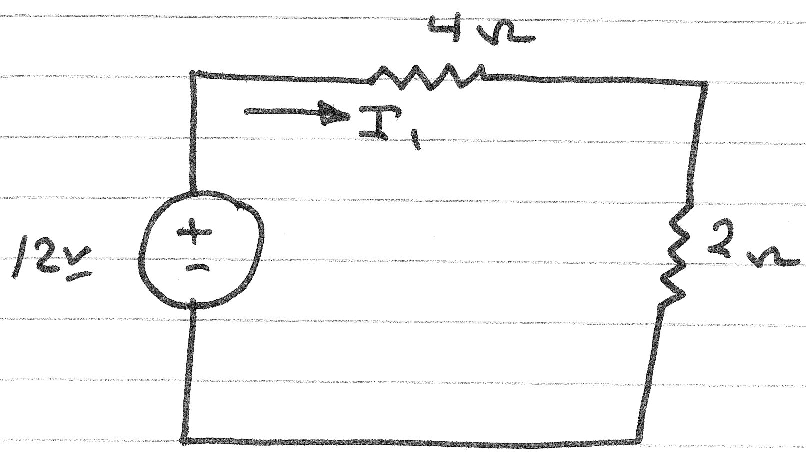
$$\therefore V_x = \frac{R_1 R_2}{R_1 + R_2} I_x$$

$$I_1 = \frac{V_x}{R_1} = \frac{R_2}{R_1 + R_2} I_x$$

$$I_2 = \frac{V_x}{R_2} = \frac{R_1}{R_1 + R_2} I_x$$

Find I_3



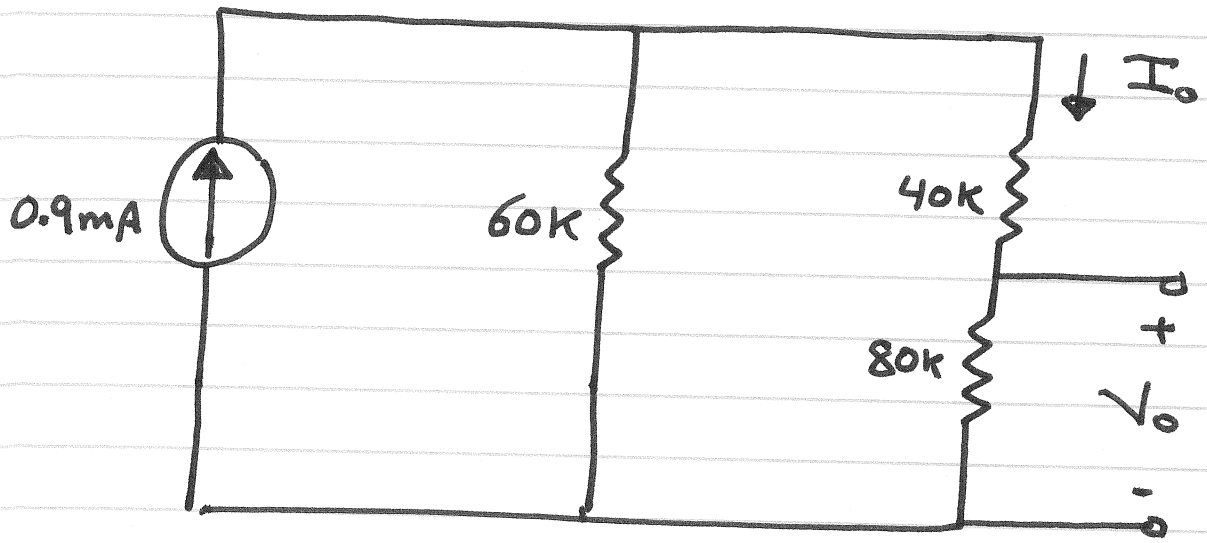


$$I_1 = \frac{12}{4+2} = 2A$$

$$I_3 = \frac{6}{6+3} I_1$$

$$I_3 = \frac{6}{9} \cdot 2 = 1.33A$$

Find V_o



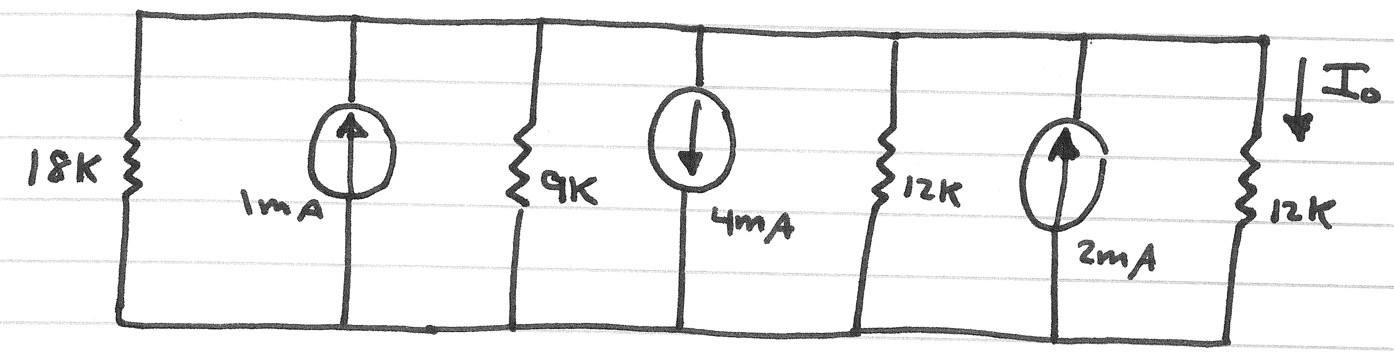
$$I_o = \frac{60K}{60K + (40K + 80K)} \, 0.9\,mA$$

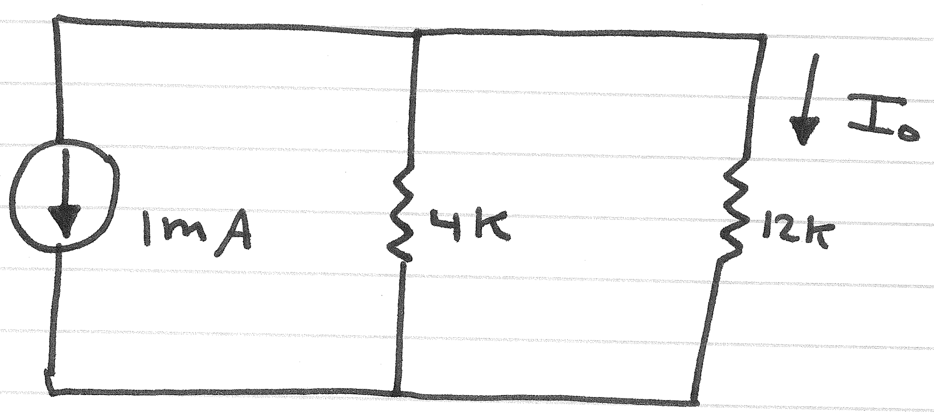
$$I_o = 0.3\,mA$$

$$V_o = 80K\, I_o$$

$$V_o = \underline{24\,V}$$

Find I_o



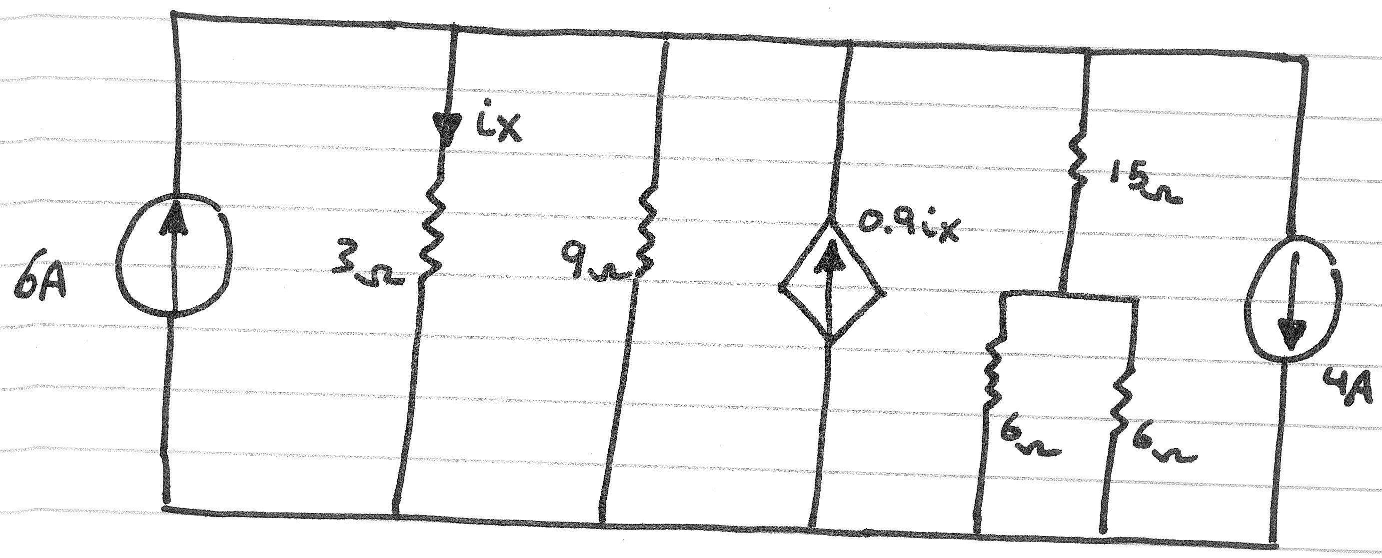


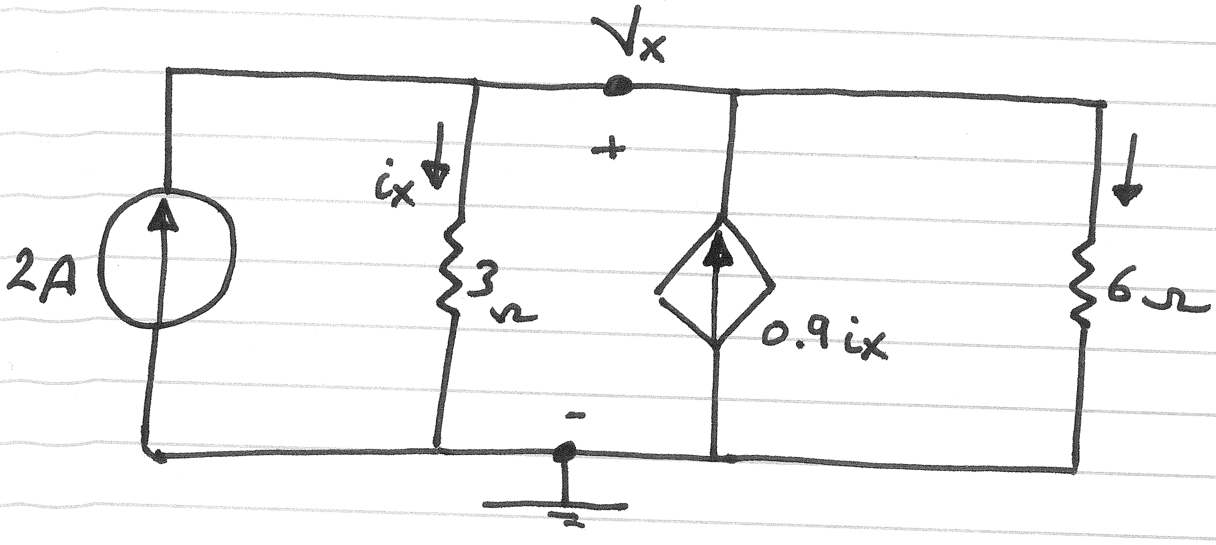
$$18K \| 9K \| 12K = 4K$$

$$I_o = -\frac{4K}{4K+12K} \, 1mA$$

$$I_o = -0.25\,mA$$

Find the power supplied by the $0.9 i_x$ source





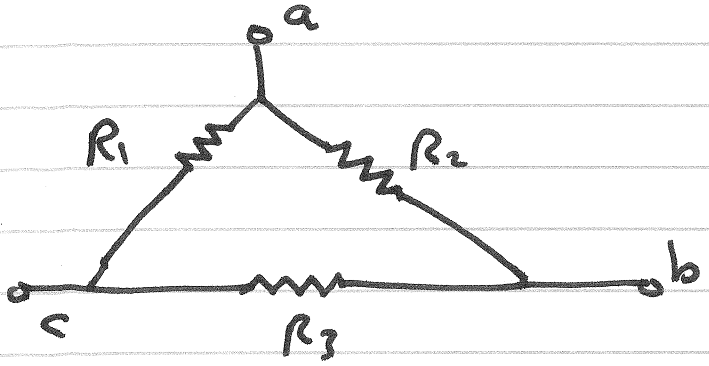
$$2 + 0.9 i_x = i_x + \frac{V_x}{6}$$

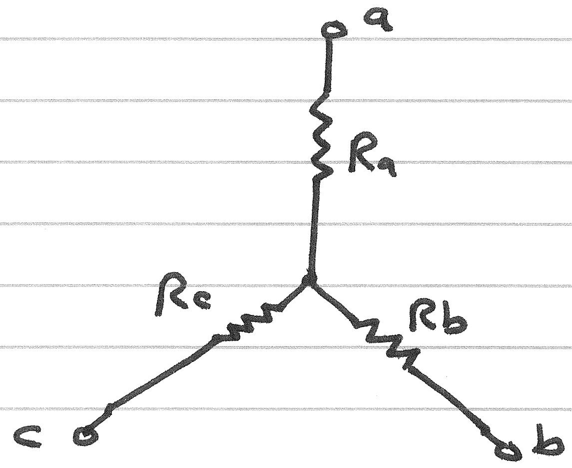
$$i_x = \frac{V_x}{3}$$

$$\therefore V_x = 10V \quad ; \quad i_x = \frac{10}{3} A$$

$$P_{0.9 i_x} = -(0.9 i_x) V_x = -30W \text{ Supplying}$$

Delta $\rightleftarrows$ Wye Transformation





$$R_{ab} = R_a + R_b = \frac{R_2(R_1+R_3)}{R_1+R_2+R_3}$$

$$R_{bc} = R_b + R_c = \frac{R_3(R_1+R_2)}{R_1+R_2+R_3}$$

$$R_{ca} = R_c + R_a = \frac{R_1(R_2+R_3)}{R_1+R_2+R_3}$$

Solving this set of equations

$$R_a = \frac{R_1 R_2}{R_1+R_2+R_3}$$

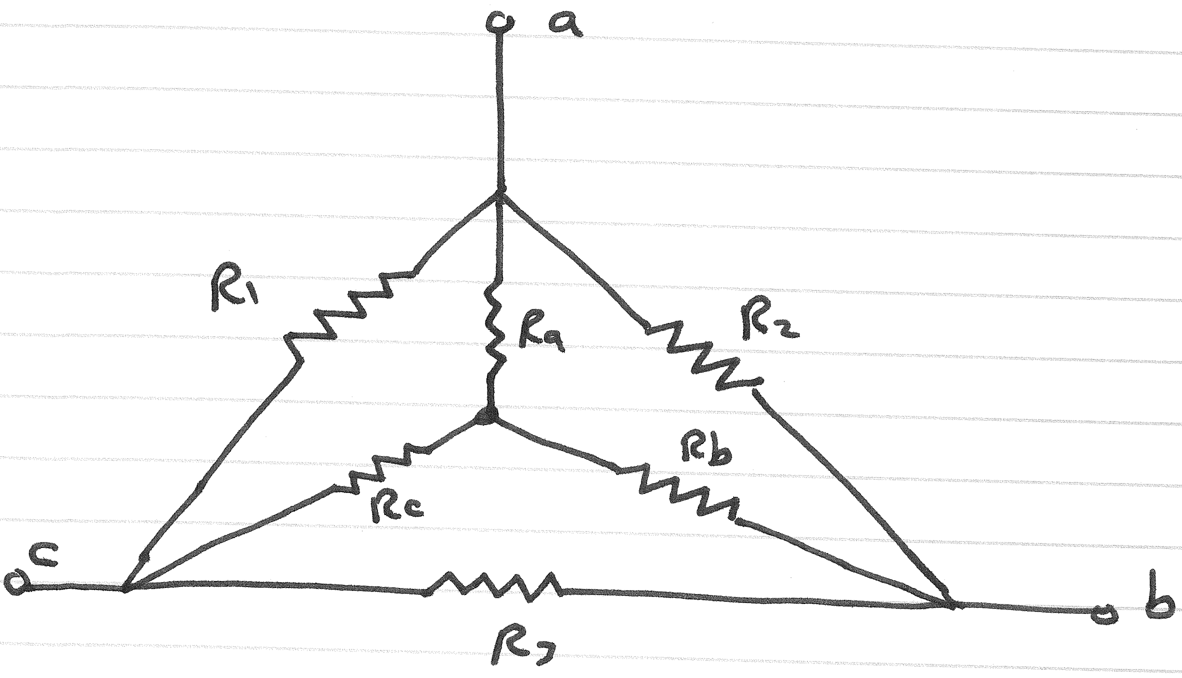
$$R_b = \frac{R_2 R_3}{R_1+R_2+R_3}$$

$$R_c = \frac{R_3 R_1}{R_1+R_2+R_3}$$

$$R_1 = \frac{R_a R_b + R_b R_c + R_c R_a}{R_b}$$

$$R_2 = \frac{R_a R_b + R_b R_c + R_c R_a}{R_c}$$

$$R_3 = \frac{R_a R_b + R_b R_c + R_c R_a}{R_a}$$



For the balanced case where

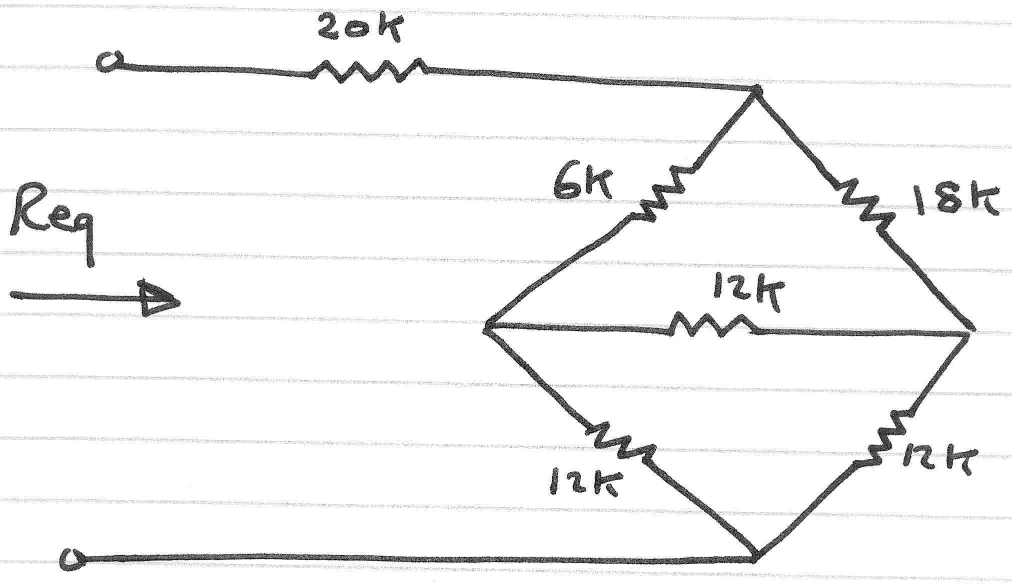
$$R_a = R_b = R_c = R_y$$

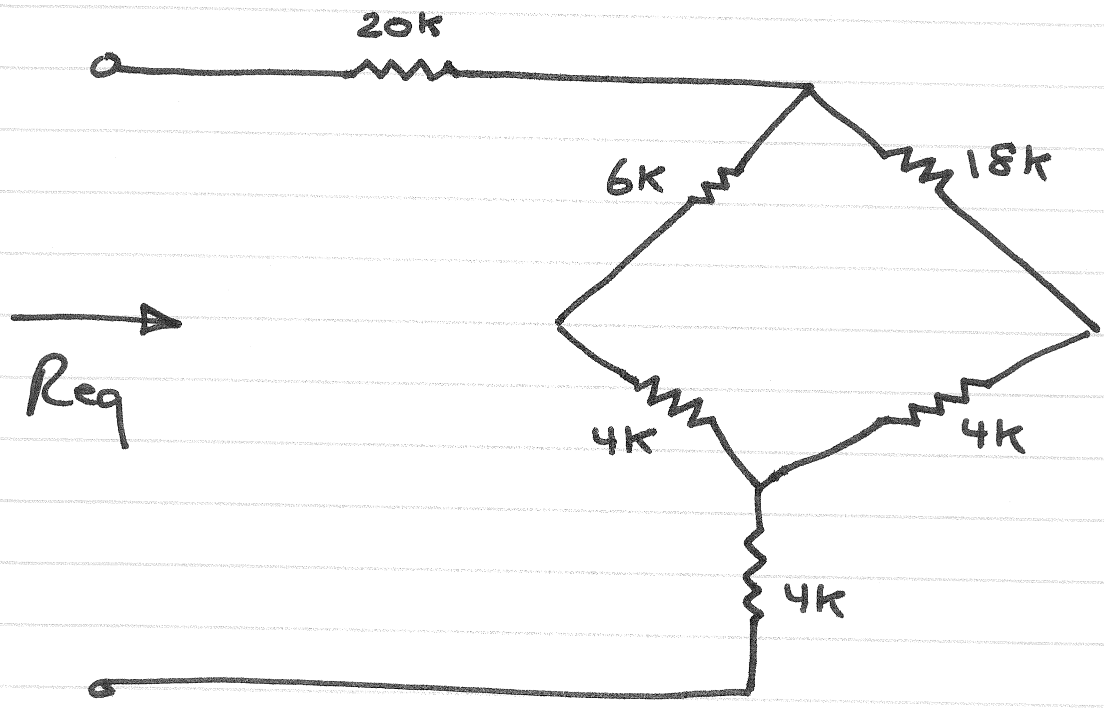
$$R_1 = R_2 = R_3 = R_\Delta$$

$$R_\Delta = 3 R_y$$

$$R_y = \frac{1}{3} R_\Delta$$

Find R_{eq}



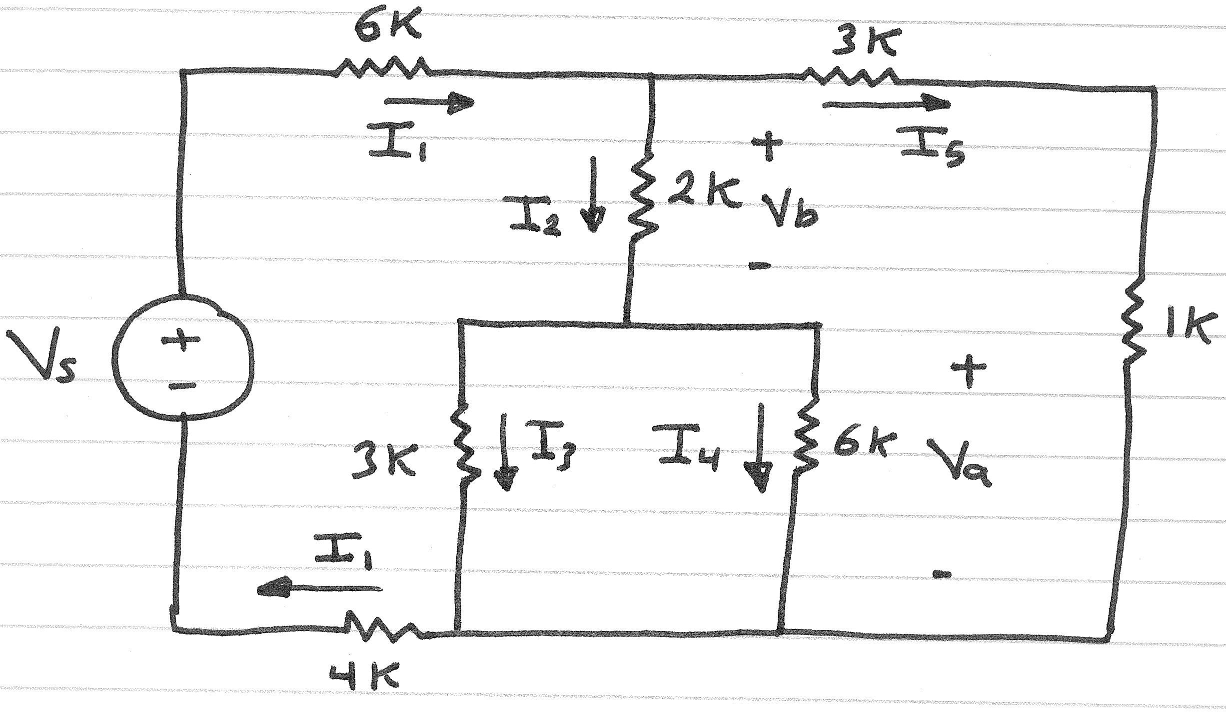


$$R_{eq} = 20k + 4k + (6k + 4k) \parallel (18k + 4k)$$

$$R_{eq} = 30.88k$$

Design :

Given $I_4 = 0.5\,mA$, Find V_s



$$V_a = (6K\Omega)(0.5\,mA) = 3\,\underline{V}$$

$$I_3 = \frac{V_a}{3K} = 1\,mA$$

$$I_2 = I_3 + I_4 = 1.5\,mA$$

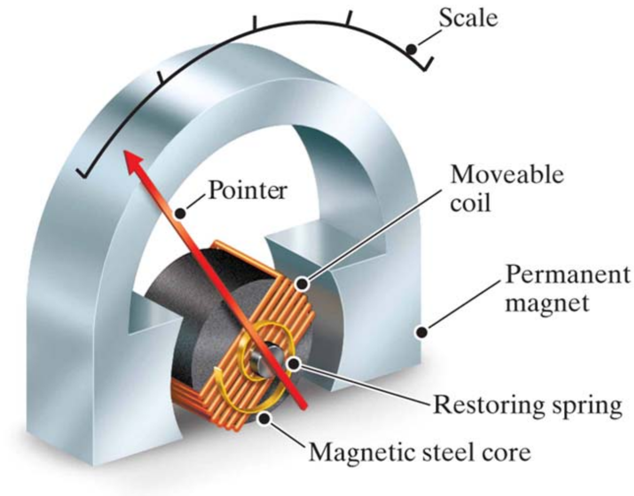
$$V_b = (2K\Omega)(1.5\,mA) = 3\,\underline{V}$$

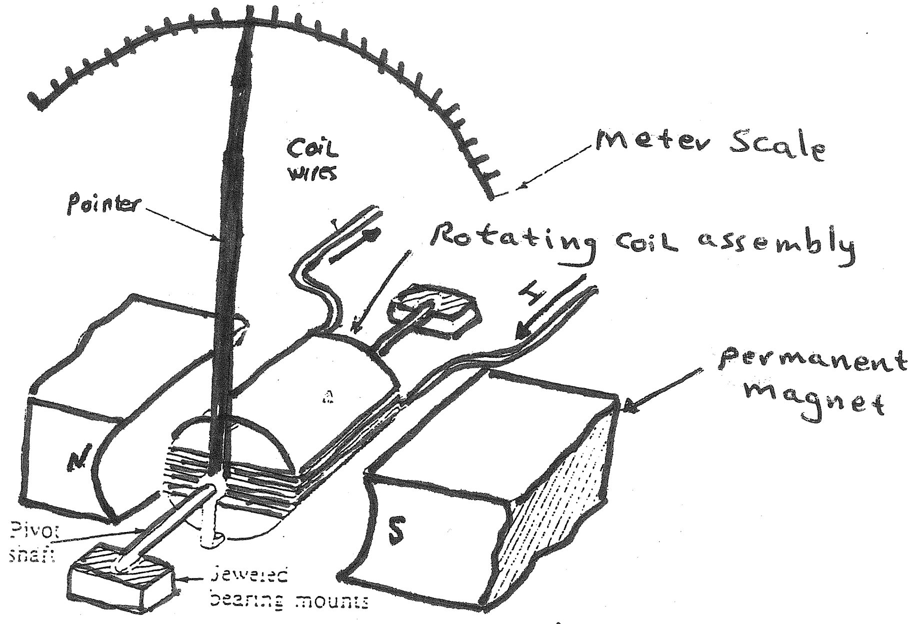
$$I_5 = \frac{V_a + V_b}{4K} = 1.5\,mA$$

$$I_1 = I_2 + I_5 = 3\,mA$$

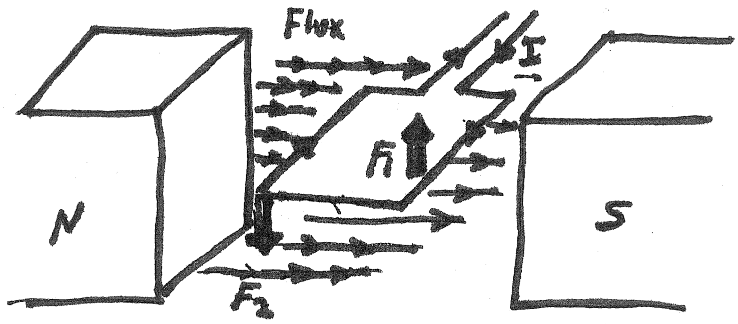
$$V_s = (10K\Omega)\,I_1 + V_b + V_a = 36\,\underline{V}$$

Figure 3.23 A schematic diagram of a d'Arsonval meter movement.





Basic components of a D'Arsonval movement



$$F_1 = F_2 = I\ell B$$

$$\tau = I\ell B d$$

If the coil has N turns

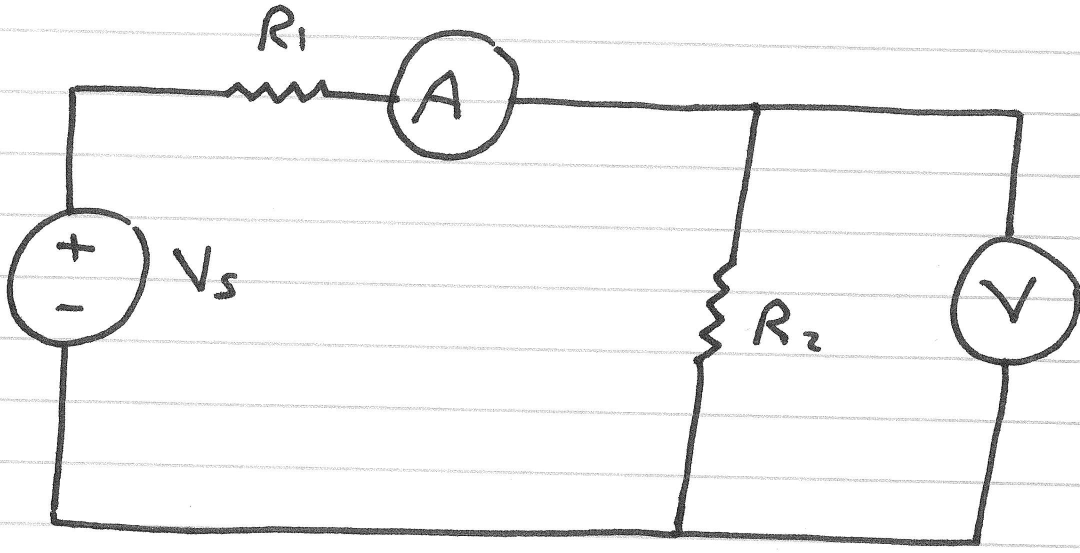
$$\tau = I\ell B N d$$

The D'Arsonval meter movement

- If a current is passed through the movable coil, the resulting magnetic field reacts with the magnetic field of the permanent magnet producing a torque which is counterbalanced by a restoring spring.

- The deflection of the pointer attached to the coil is proportional to the current produced by the quantity being measured.

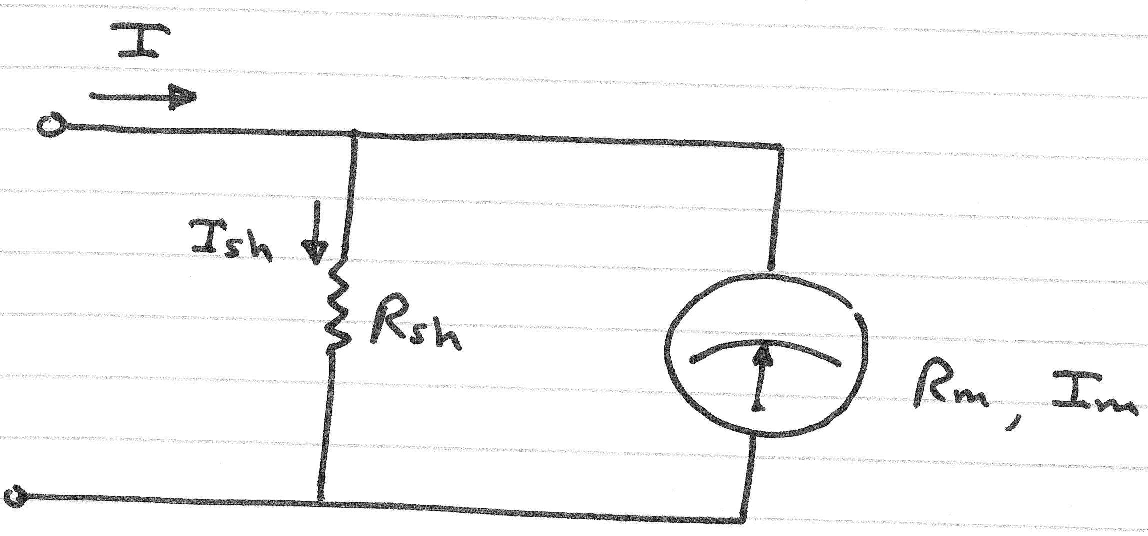
Measuring Voltage and Current



Ammeter : designed to measure current

Voltmeter : designed to measure voltage

DC Ammeter



$$R_{sh} I_{sh} = R_m I_m$$

$$R_{sh} = \frac{R_m I_m}{I_{sh}} = \frac{R_m I_m}{I - I_m}$$

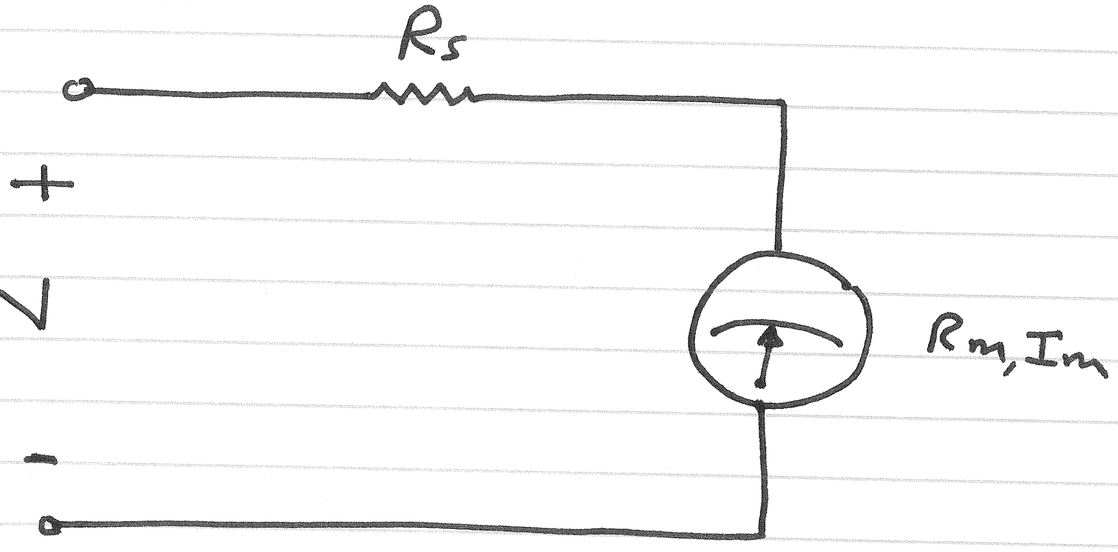
A 0-1mA meter movement with an internal resistance of 100 Ω is to be converted to a 0-100mA Ammeter.

$I_m = 1mA$
$R_m = 100\ \Omega$

$I = 100mA$
$I_{sh} = 99\ mA$

$$R_{sh} = \frac{I_m R_m}{I - I_m} = 1.01\ \Omega$$

Dc Voltmeter



$$V = R_s I_m + R_m I_m$$

$$R_s = \frac{V - R_m I_m}{I_m}$$

A basic D'Arsonval movement with $I_m = 1mA$ and $R_m = 100\,\Omega$ is to be converted into a dc voltmeter with the range 0 - 10 V

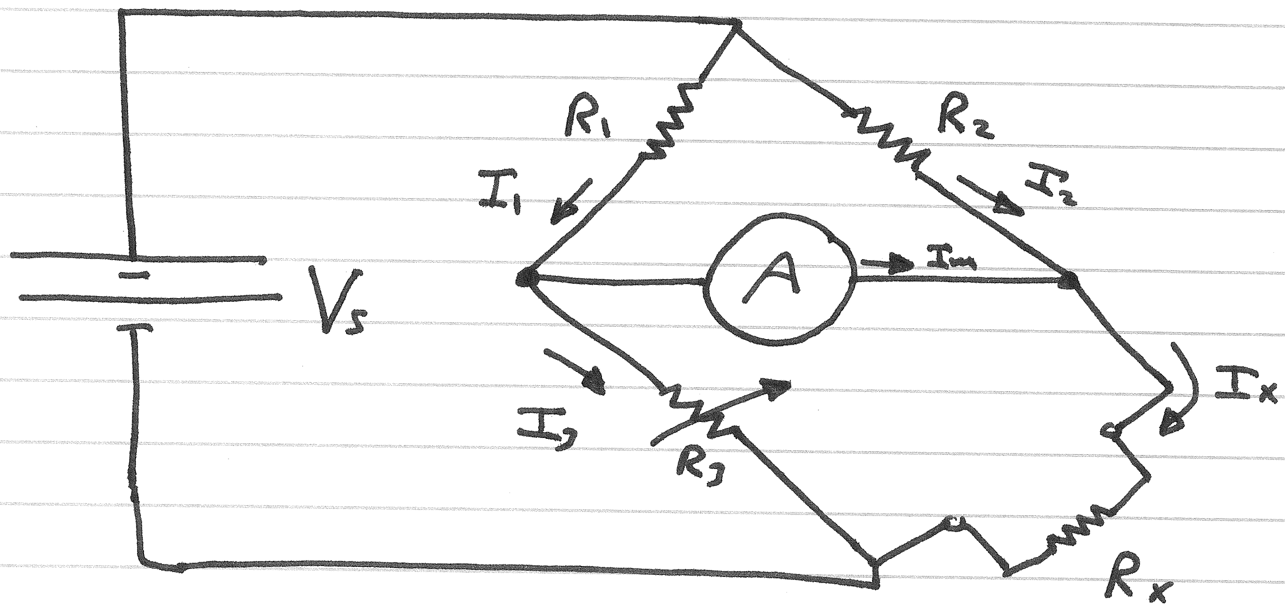
$$R_s = \frac{V - R_m I_m}{I_m}$$

$$= \frac{10 - (100)(1 \times 10^{-3})}{1 \times 10^{-3}}$$

$$R_s = 9900\,\Omega$$

Measuring Resistance

Wheatstone Bridge



R_3 is adjusted until $I_m = 0$

Bridge is balanced : $I_1 = I_3$
$I_2 = I_x$
$V_m = 0$

$$R_1 I_1 = R_2 I_2$$

$$R_3 I_3 = R_x I_x$$

$$\frac{R_1 I_1}{R_3 I_3} = \frac{R_2 I_2}{R_x I_x}$$

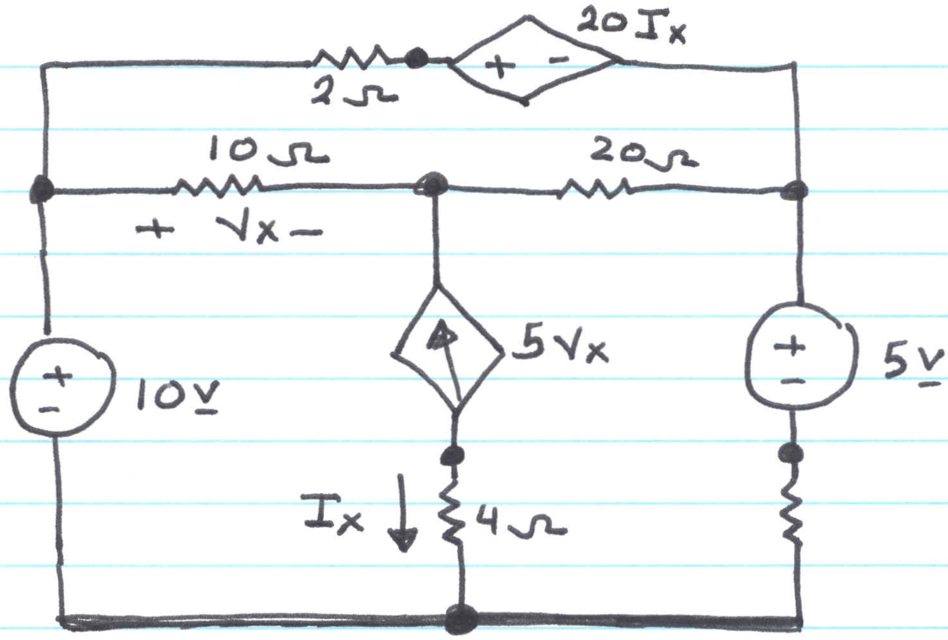
$$\frac{R_1}{R_3} = \frac{R_2}{R_x} \longrightarrow R_x = \frac{R_2 R_3}{R_1}$$

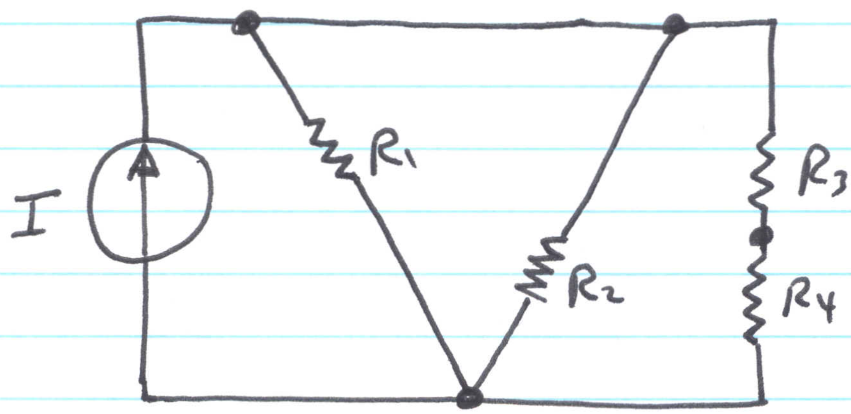
Voltage and Current Laws

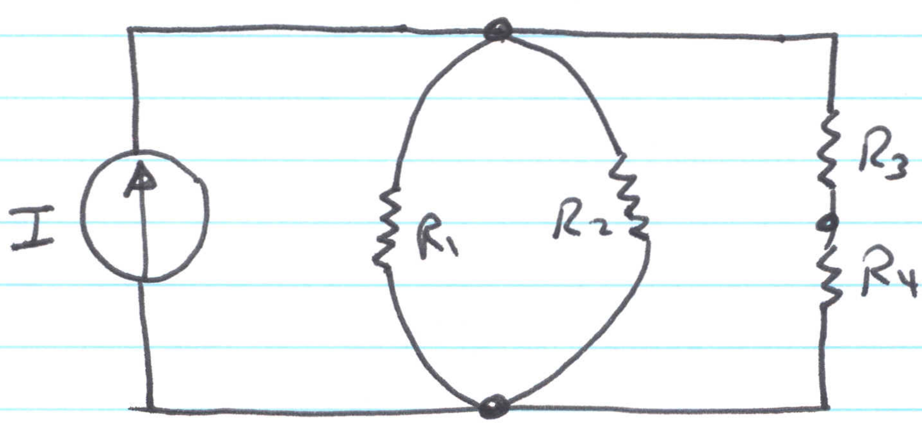
Node : A point of Connection of two or more Circuit elements.

Loop : Any closed path through the Circuit in which no node is crossed more than once

Mesh : Any Loop that does not Contain within it a nother Loop

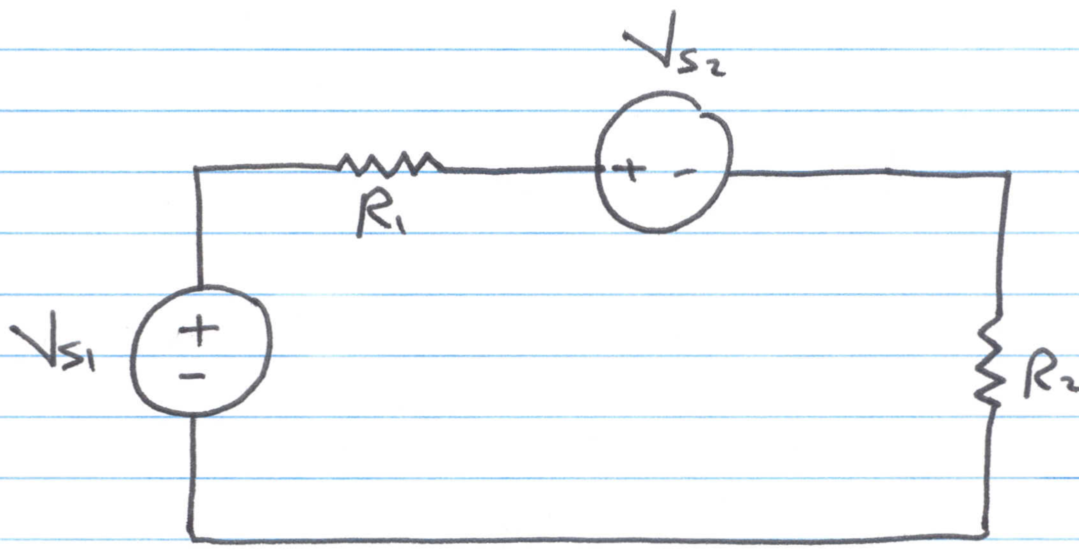

20 Ix
2Ω
10Ω
20Ω
+ Vx −
5Vx
10V
5V
Ix
4Ω


I
R1
R2
R3
R4


I
R1
R2
R3
R4

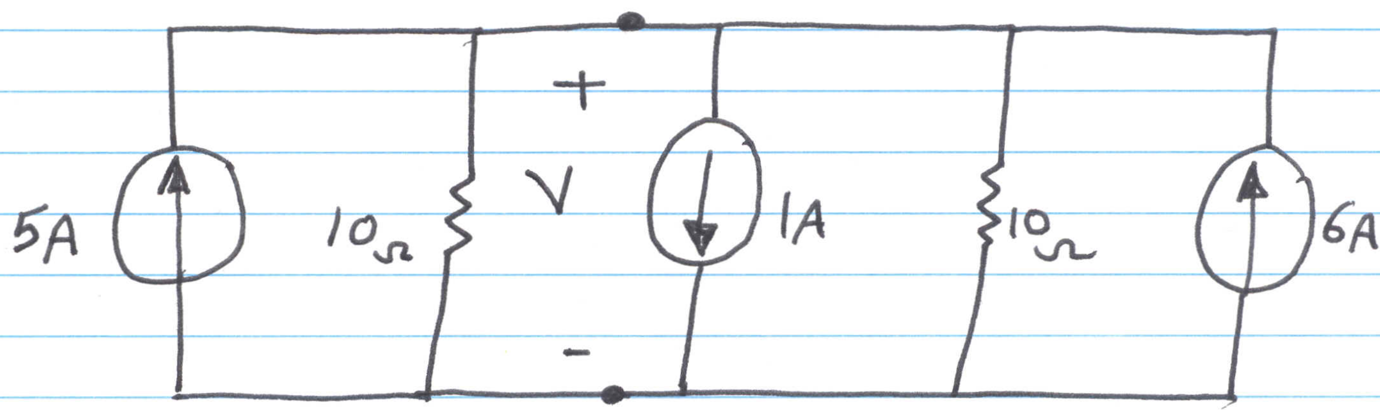
Series Connections

All of the elements in a circuit that carry the same current are said to be connected in series



Parallel Connections

Elements in a Circuit having a Common Voltage across them are said to be Connected in Parallel.

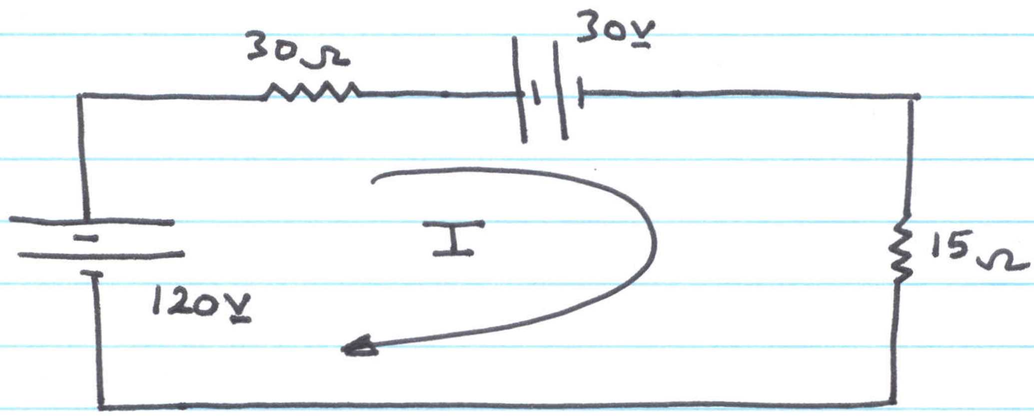


Kirchhoff's Voltage Law : KVL

KVL : The algebraic sum of the voltages around any Loop is zero.

Analysis of a single-Loop Circuit

Find I



$$30I + 30 + 15I - 120 = 0$$

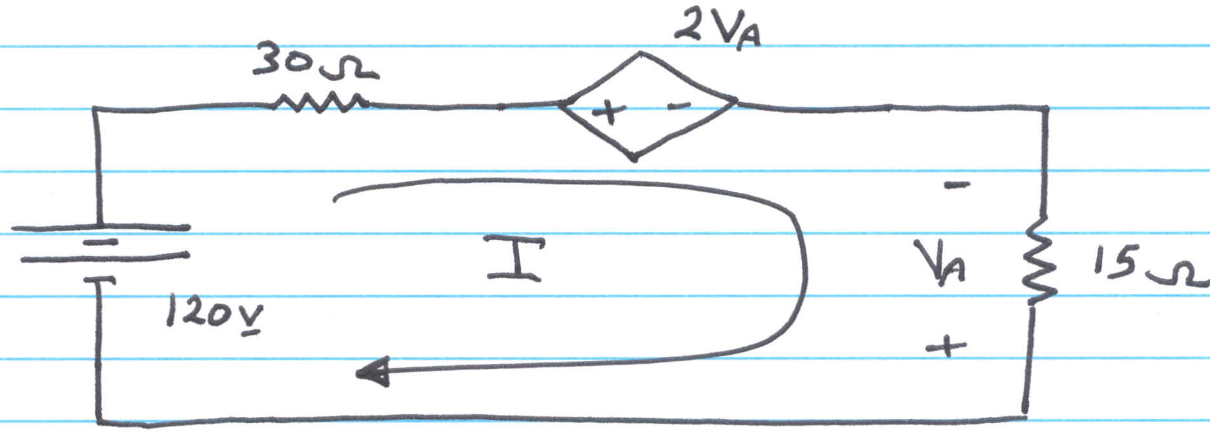
$\therefore \quad I = 2A$

$V_{30\Omega} = 60\,V$

$V_{15\Omega} = 30\,V$

Analysis of a Circuit containing a dependent source

Find I



$$30I + 2V_A + 15I - 120 = 0$$

$$V_A = -15I$$

$$\therefore \quad I = 8A$$

$$V_A = -120V$$

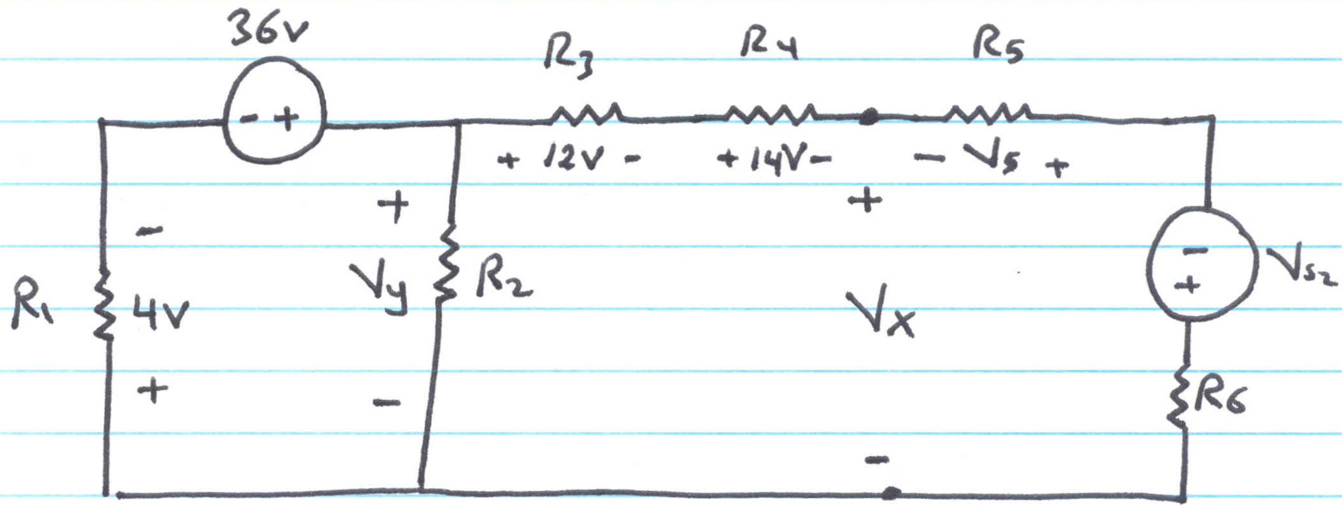
* Calculate the power absorbed by each circuit element

Answers:

$P_{120V} = -960W$, $P_{30\Omega} = 1920W$

$P_{2V_A} = -1920W$, $P_{15\Omega} = 960W$

Applying KVL



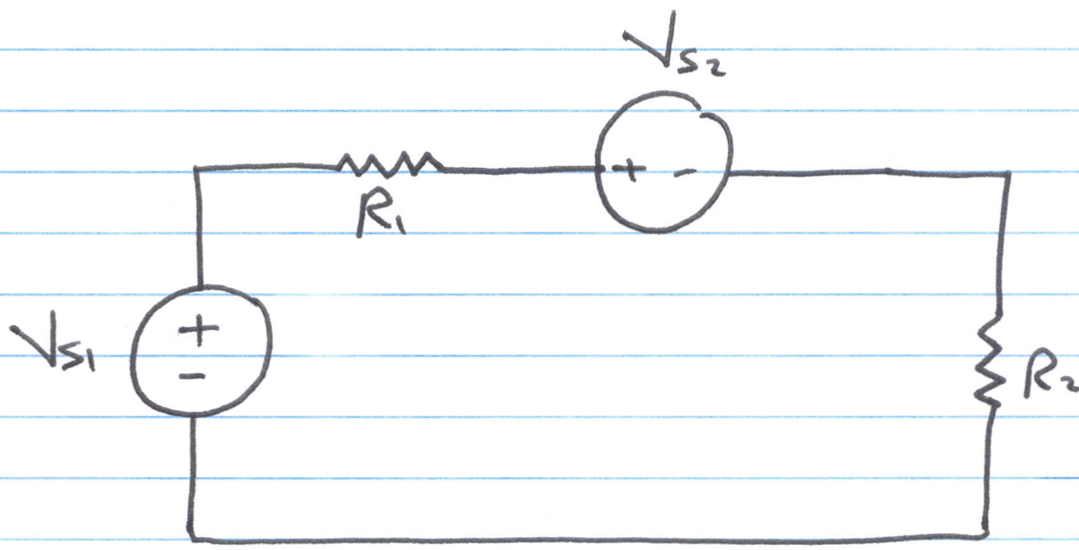
Find V_x and V_y

$V_y = 36 - 4 = 32\ \underline{V}$

$V_x = -14 - 12 + 32 = 6\ \underline{V}$

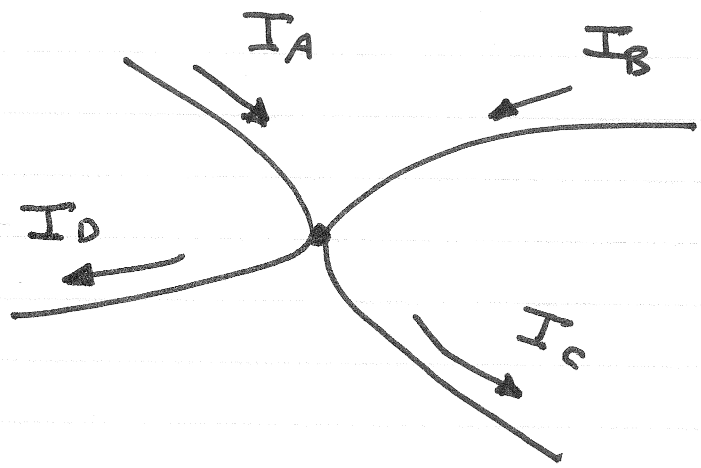
Series Connections

All of the elements in a circuit that carry the same current are said to be connected in series



Kirchhoff's Current Law : KCL

KCL : The algebraic sum of the current entering any node is zero



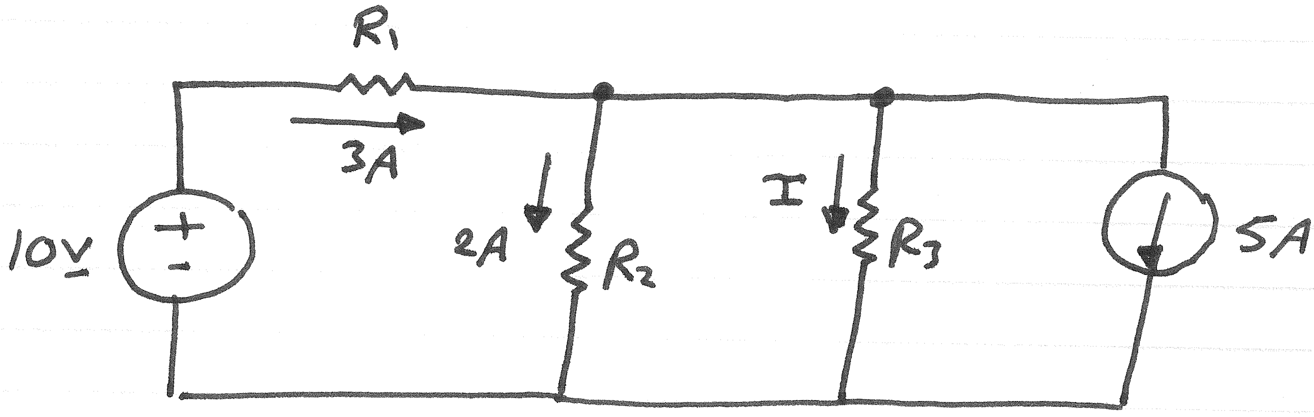
$$I_A + I_B - I_C - I_D = 0$$

KCL : Alternative Form

Current In = Current OUT

$$I_A + I_B = I_C + I_D$$

KCL Application



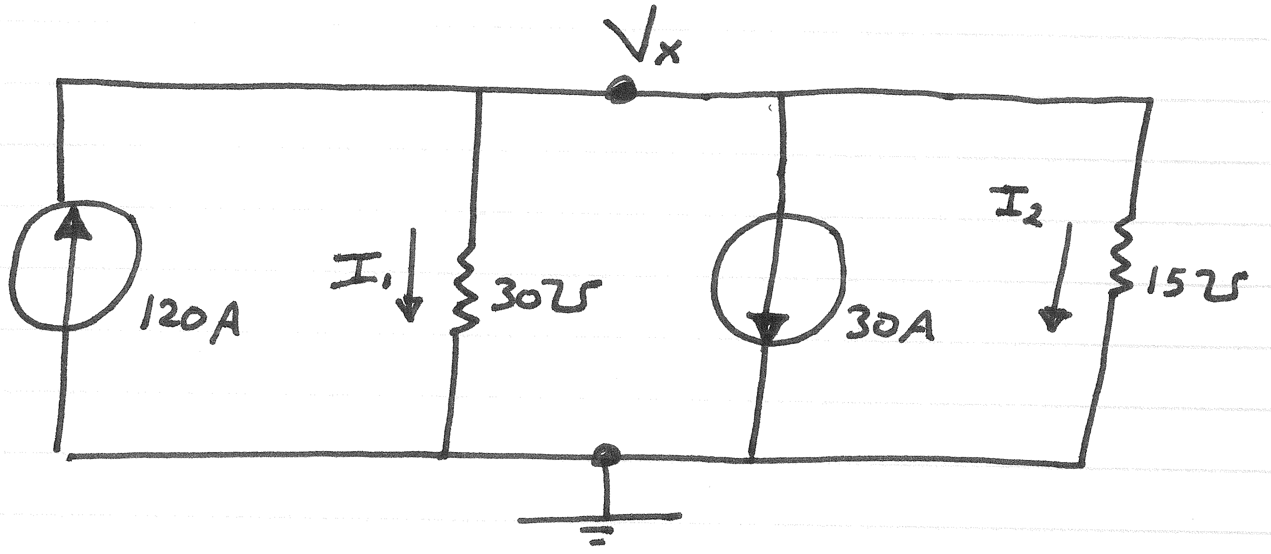
Find I

KCL : $3 = 2 + I + 5$

$\therefore I = -4A$

The single node-pair Circuit

Find V_x



$I = GV$

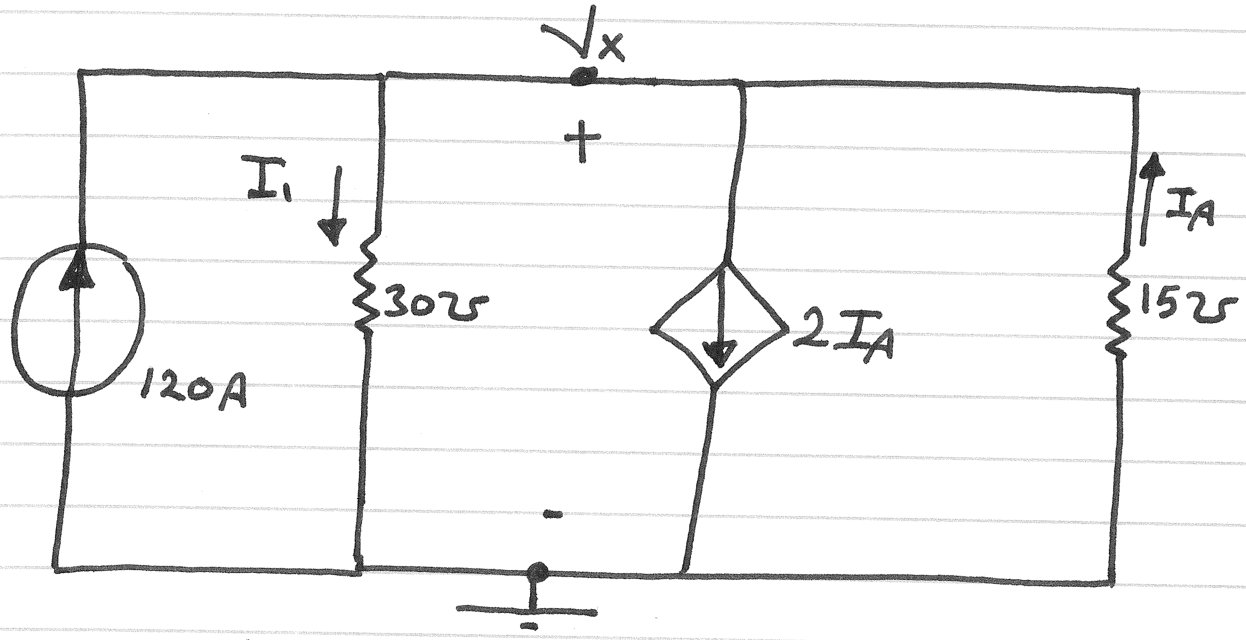
KCL : $120 = 30V_x + 30 + 15V_x$

$\therefore V_x = 2\,\underline{V}$

$\therefore I_{30℧} = 60\,A$

$I_{15℧} = 30\,A$

Analysis of Circuit Containing dependent sources



Find V_x

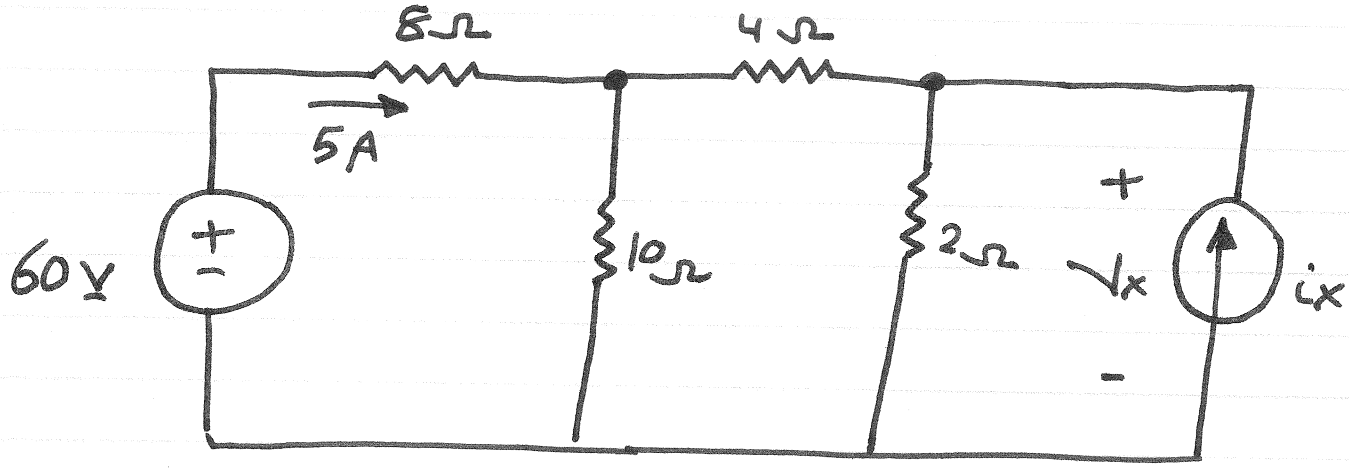
KCL : $120 + I_A = I_1 + 2I_A$

$I_A = -15V_x$

$I_1 = 30V_x$

$\therefore V_x = 8\underline{V}$

Applying KVL and KCL

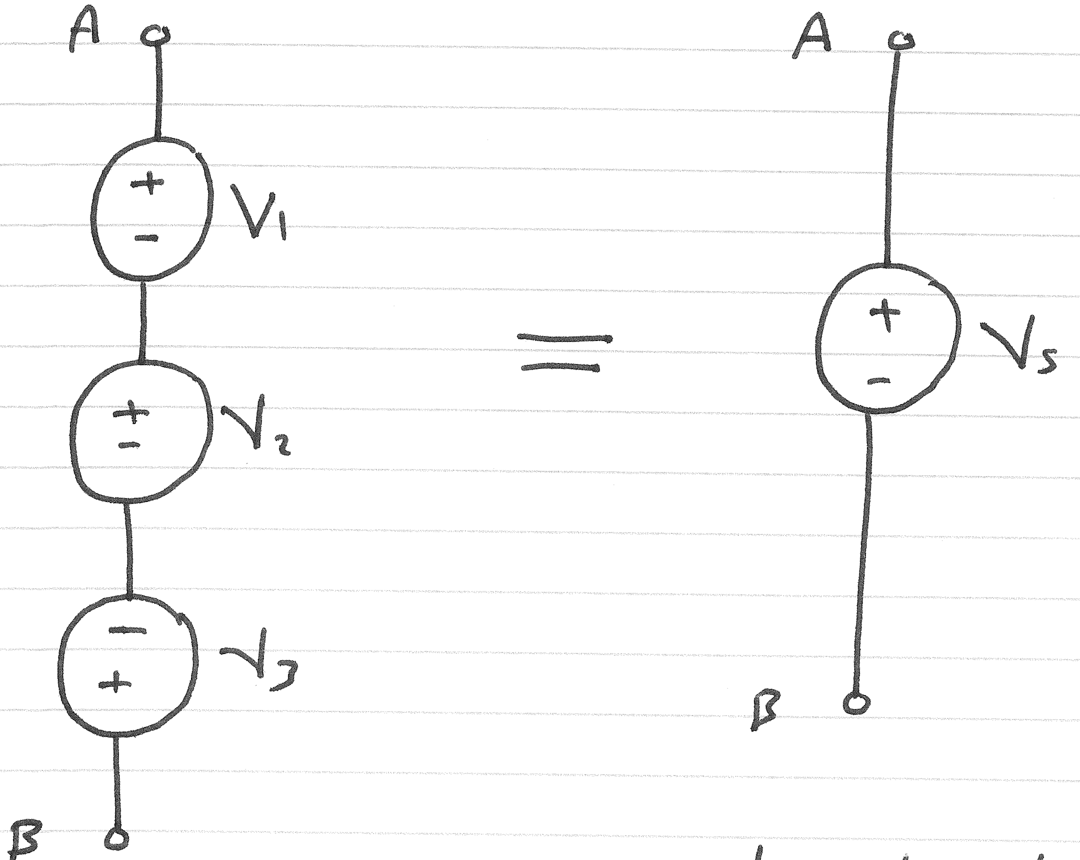


Solve for V_x and i_x

Answer : $V_x = 8V$ and $i_x = 1A$

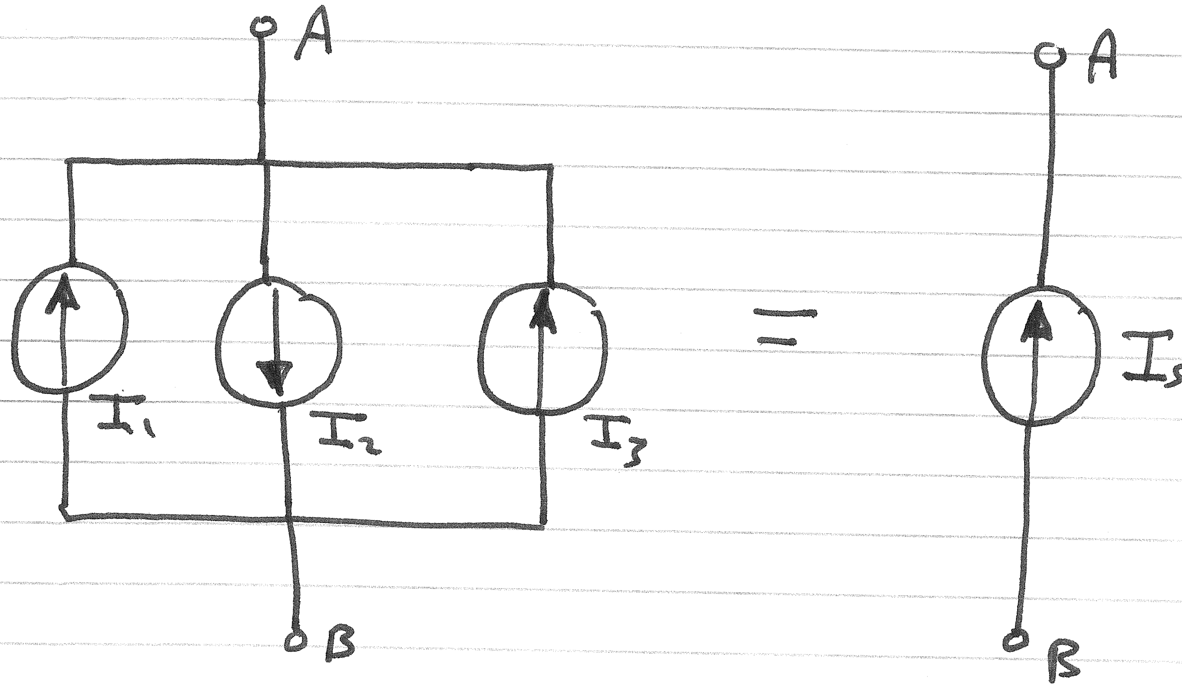
Series and Parallel Sources

Voltage sources connected in series can be combined into an equivalent source:



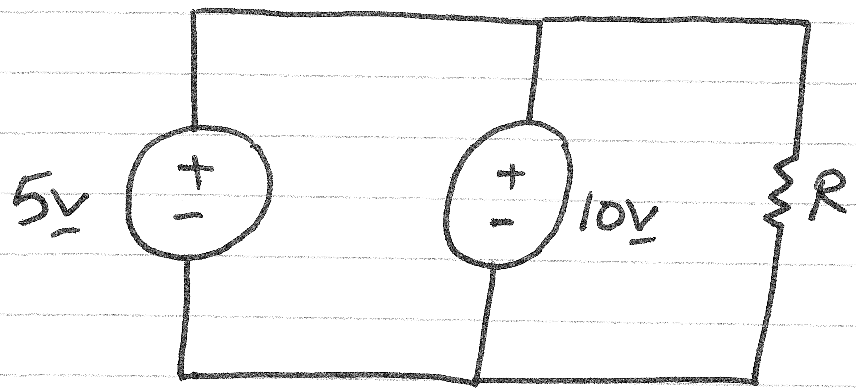
$$V_s = V_1 + V_2 - V_3$$

Current sources connected in parallel can be combined into an equivalent current source :

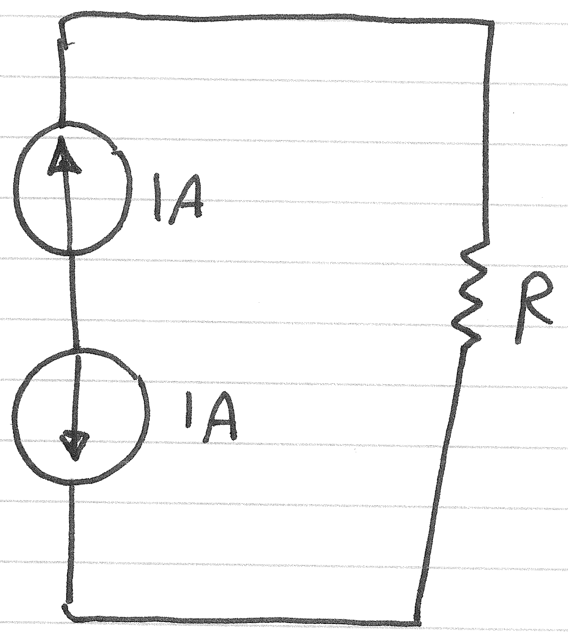


$$I_s = I_1 + I_3 - I_2$$

Impossible Circuits

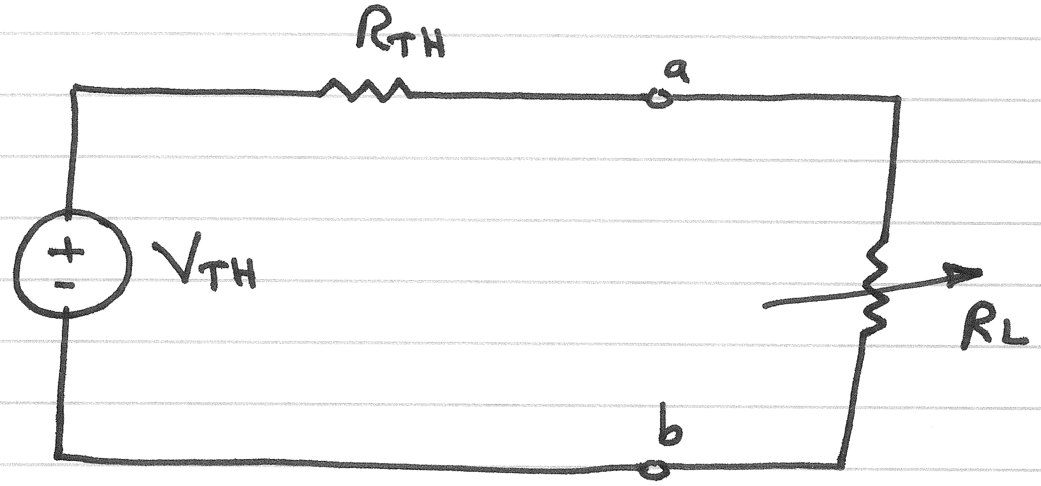


X



X

Maximum Power Transfer



A load resistance will recieve maximum power from a circuit when the resistance of the load is exactly the same as the thevenin's resistance looking back at the circuit.

$$R_L = R_{TH}$$

$$P_L = \frac{V_L^2}{R_L}$$

$$V_L = \frac{R_L}{R_L + R_{TH}} \cdot V_{TH}$$

$$P_L = \frac{V_{TH}^2 R_L}{(R_L + R_{TH})^2}$$

$$\frac{d P_L}{d R_L} = \frac{V_{TH}^2 \left((R_L + R_{TH})^2 - 2 R_L (R_{TH} + R_L) \right)}{(R_L + R_{TH})^4}$$

for $\frac{d P_L}{d R_L} = 0$

$$(R_L + R_{TH})^2 - 2 R_L (R_{TH} + R_L) = 0$$

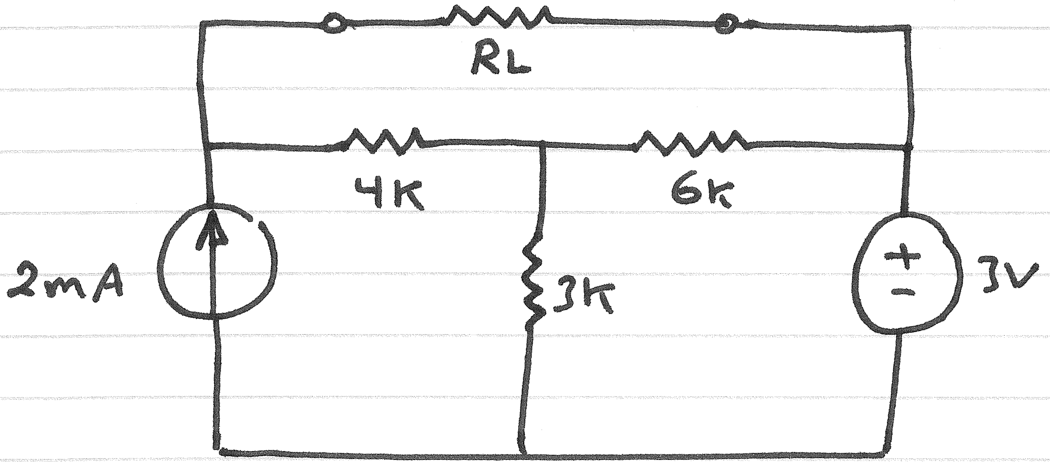
$$(R_L + R_{TH}) \left((R_{TH} + R_L) - 2 R_L \right) = 0$$

$$\therefore \quad R_{TH} - R_L = 0$$

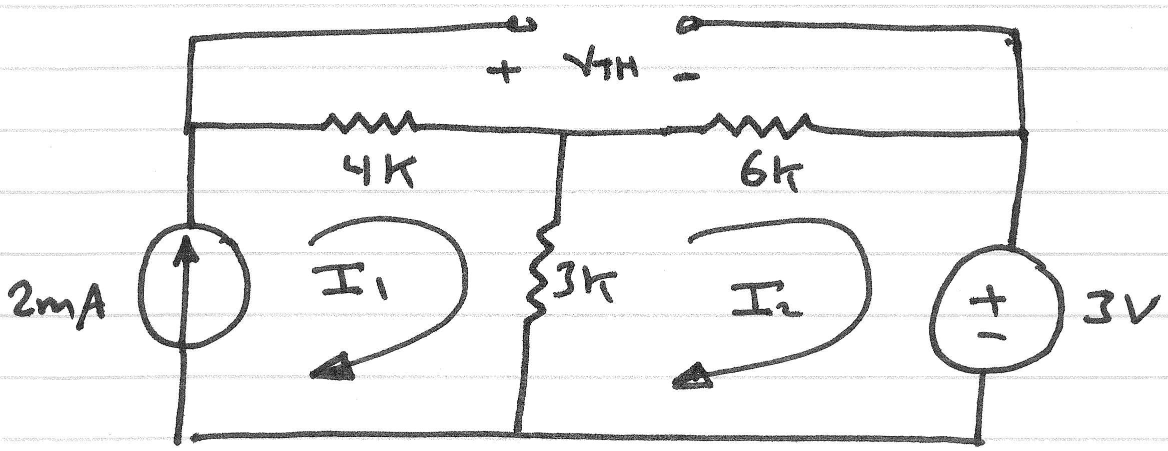
$$\therefore \quad R_L = R_{TH}$$

$$P_{L,max} = \frac{V_{TH}^2}{4 R_L} = \frac{V_{TH}^2}{4 R_{TH}}$$

- Find the value of R_L for maximum power transfer in the circuit shown.
- Find the maximum power



To find V_{TH}



$I_1 = 2mA$ Constrain equation

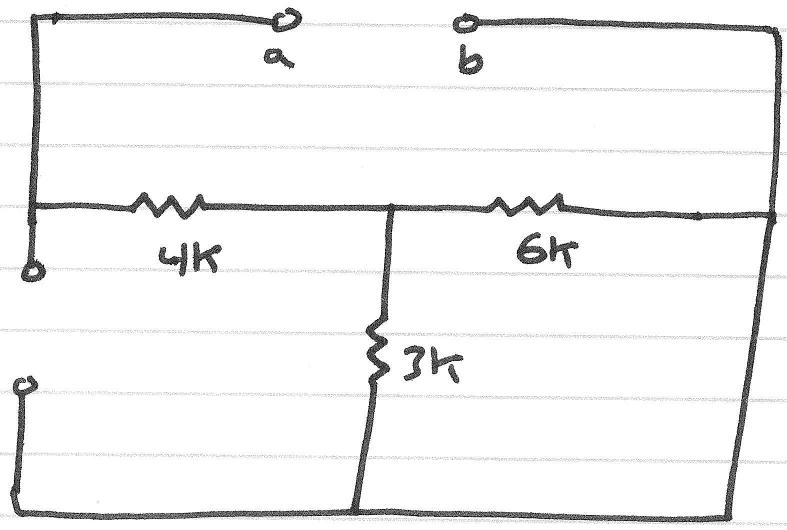
$-3 = 9K\, I_2 - 3K\, I_1$

$\therefore\; I_2 = \frac{1}{3} mA$

$V_{TH} = 4k\, I_1 + 6k\, I_2$

$V_{TH} = 10V$

To find R_{TH}



$R_{TH} = 4K + 3K \| 6K$

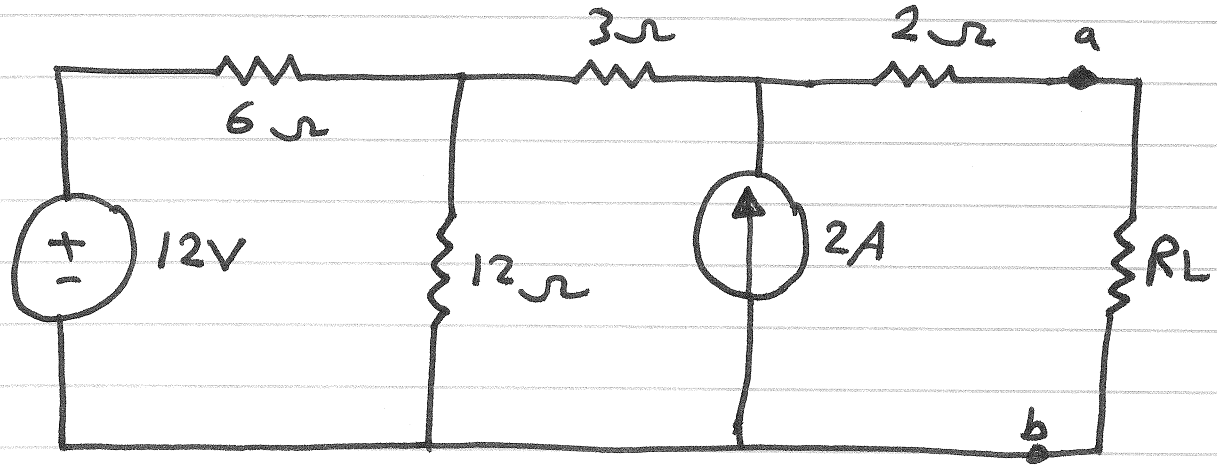
$= 4K + 2K$

$R_{TH} = 6K$

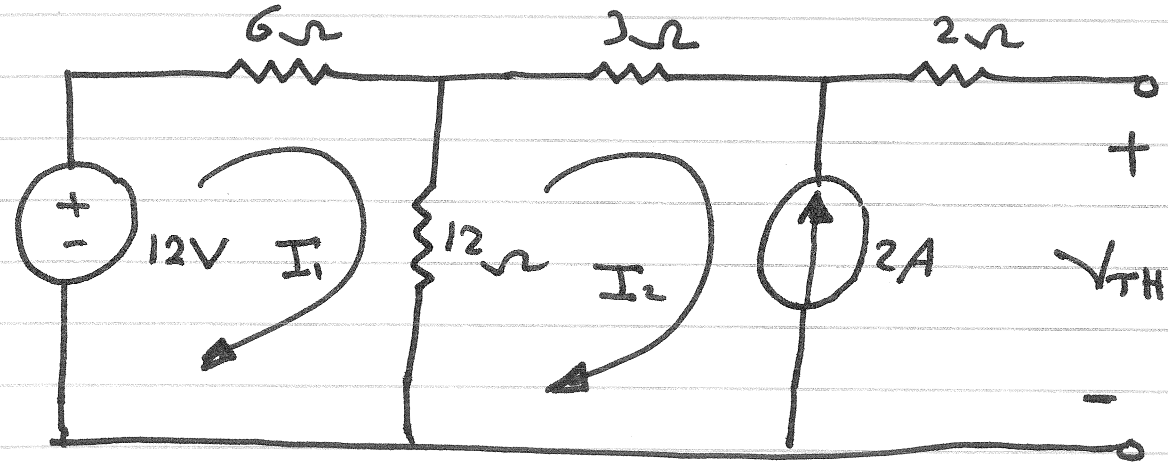
$\therefore R_L = R_{TH} = 6K$

$$P_{L,max} = \frac{V_{TH}^2}{4R_{TH}} = \frac{25}{6} \text{ mW}$$

- Find the value of R_L for maximum power transfer in the circuit shown
- Find the maximum power



1) To find V_{TH}



$I_2 = -2A$ Constrain equation

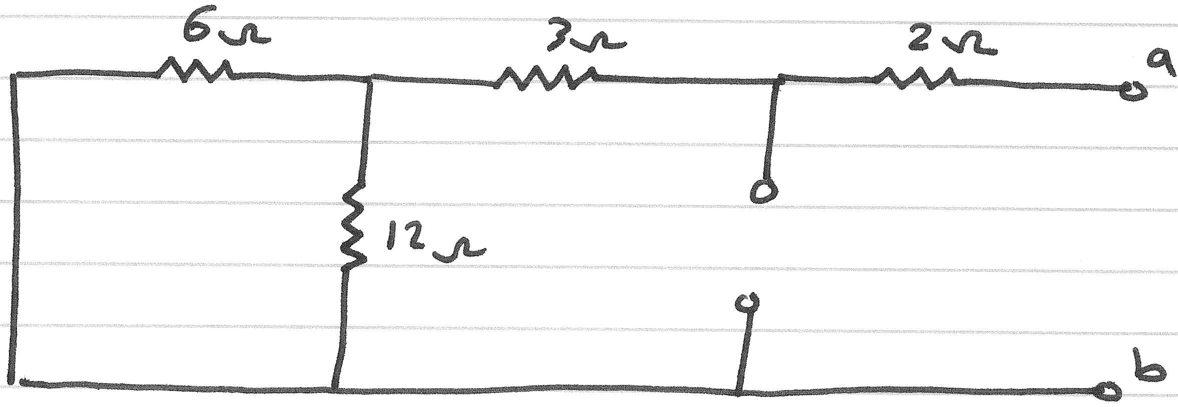
$12 = 18I_1 - 12I_2$

$\therefore I_1 = -\frac{2}{3}A$

$$V_{TH} = -3I_2 - 6I_1 + 12$$

$$\therefore V_{TH} = 22V$$

2) To find R_{TH}



$$R_{TH} = 2 + 3 + 6 \parallel 12$$

$$R_{TH} = 2 + 3 + 4 = 9\Omega$$

$$\therefore R_L = R_{TH} = 9\Omega$$

$$\therefore P_{L,max} = \frac{V_{TH}^2}{4R_{TH}} = 13.44W$$

Mesh Analysis

1. Mesh analysis: another method for analyzing circuits, applicable to **planar** circuits.

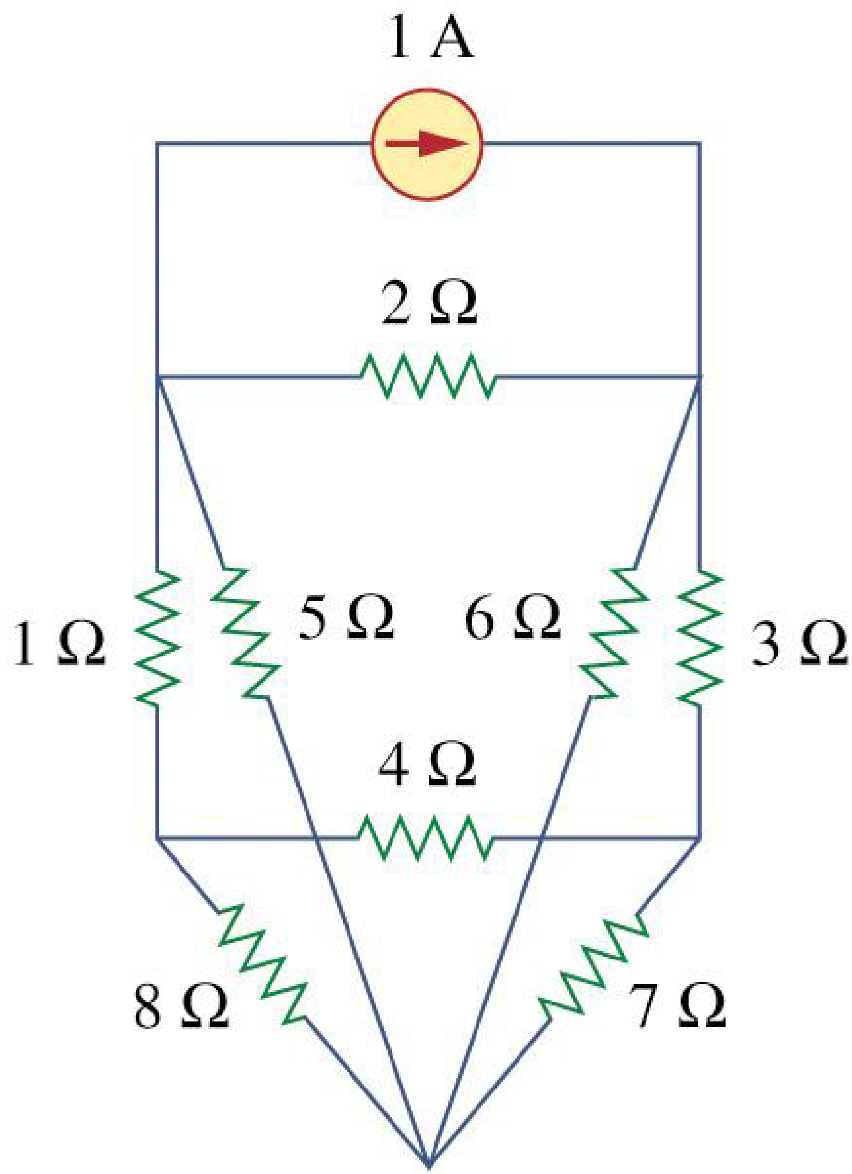
2. A Mesh is a loop which does not contain any other loops within it.

3. Nodal analysis applies KCL to find voltages in a given circuit, while Mesh Analysis applies **KVL** to calculate unknown currents.

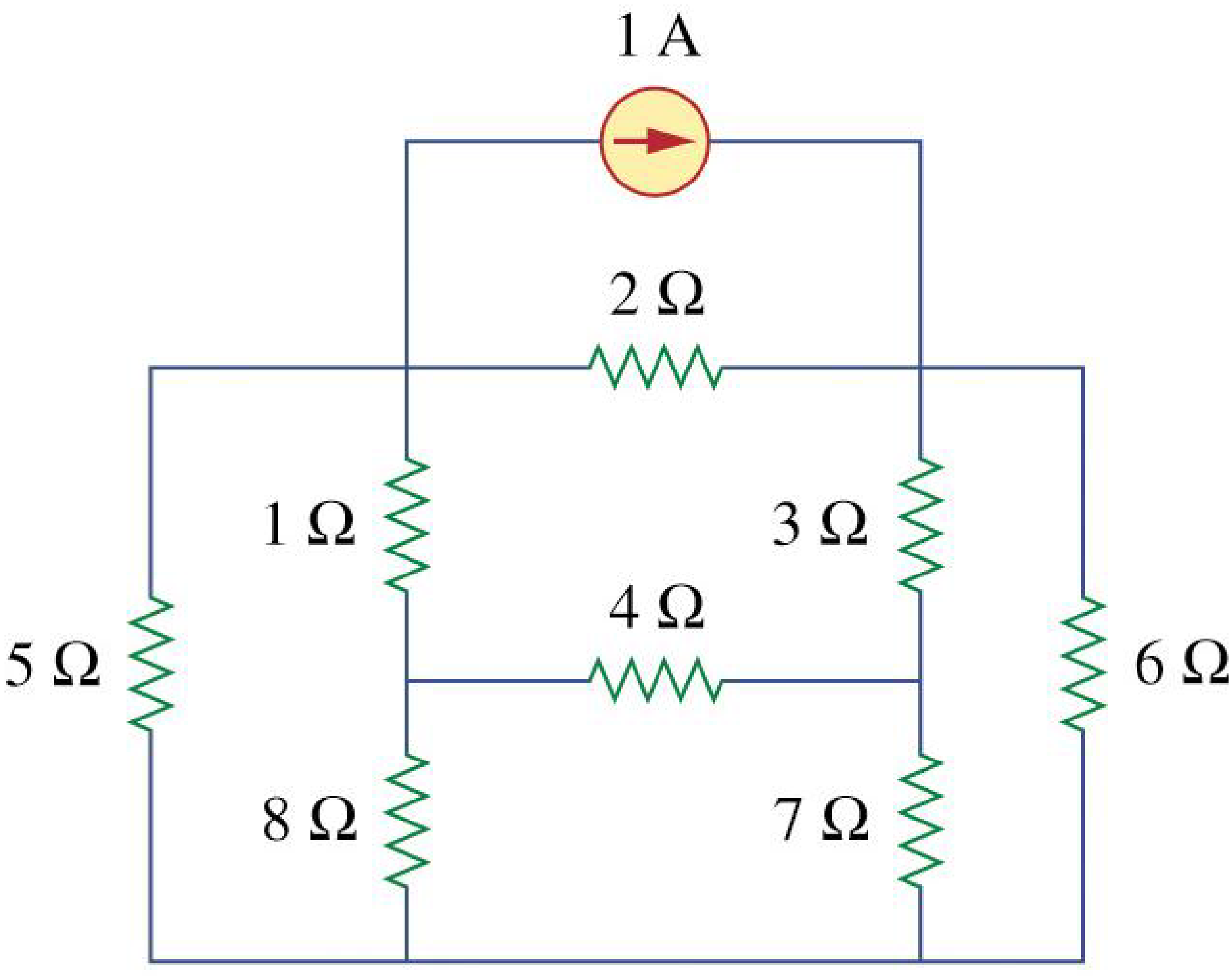
Mesh Analysis

A circuit is **planar** if it can be drawn on a plane with no branches crossing one another. Otherwise it is non planar.

The circuit in (a) is planar, because the same circuit that is redrawn(b) has no crossing branches



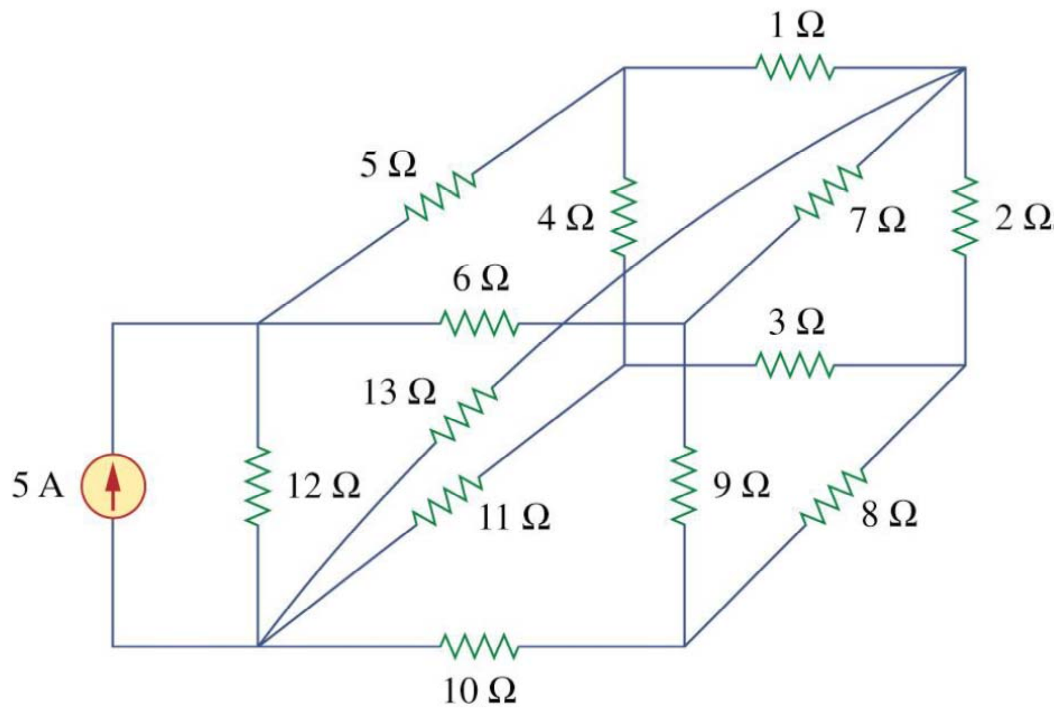
(a)



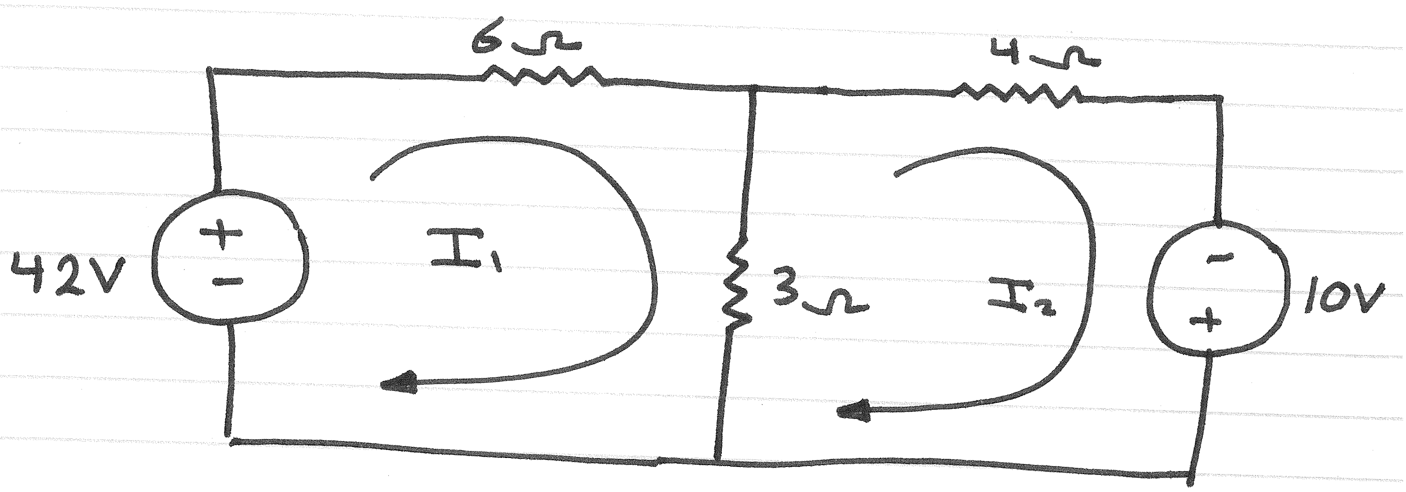
(b)

Mesh Analysis

A **non planar** circuit.



Mesh Analysis



KVL for mesh ① :

$$42 = 6I_1 + 3(I_1 - I_2)$$

$$42 = 9I_1 - 3I_2$$

KVL for mesh ② :

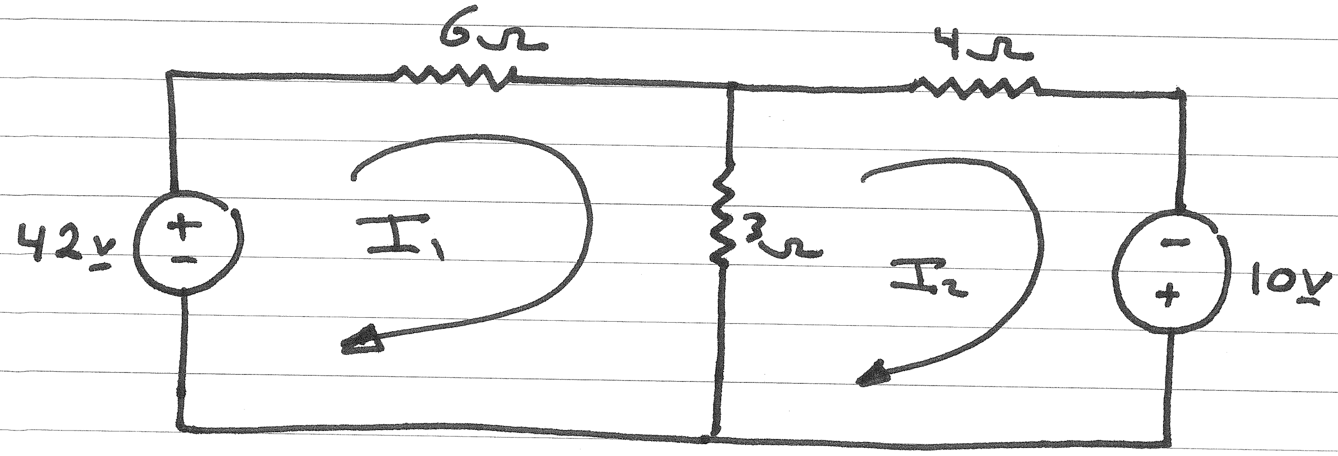
$$+10 = 4I_2 + 3(I_2 - I_1)$$

$$10 = -3I_1 + 7I_2$$

$$\therefore \quad I_1 = 6A$$

$$I_2 = 4A$$

Mesh Analysis



KVL for mesh ①:

$$42 = 9I_1 - 3I_2$$

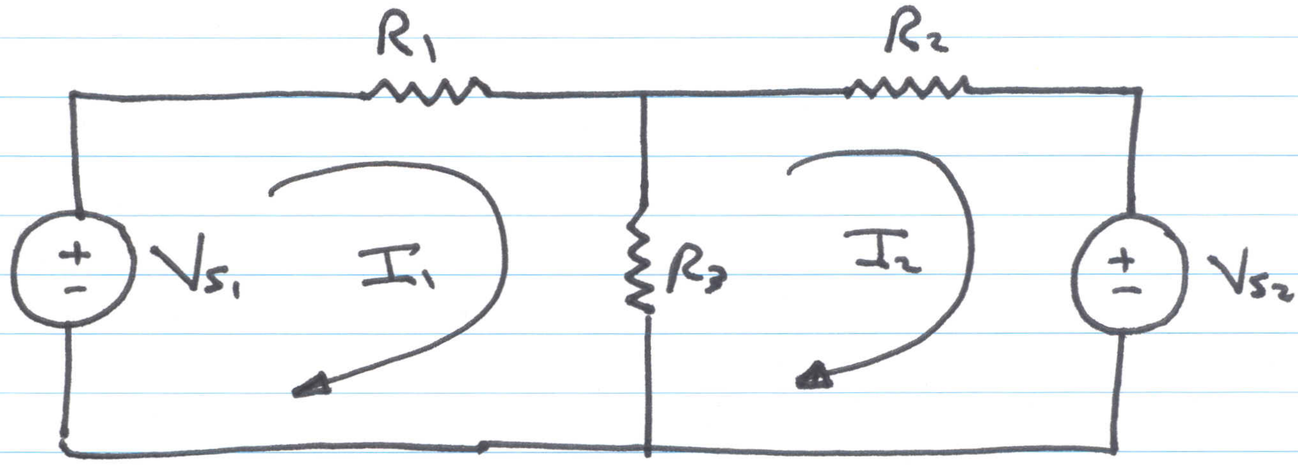
KVL for mesh ②:

$$10 = -3I_1 + 7I_2$$

$\therefore$ $I_1 = 6A$

$I_2 = 4A$

Mesh Analysis



Applying KVL for mesh 1 ;

$$-V_{s_1} + R_1 I_1 + R_3 (I_1 - I_2) = 0$$

$$V_{s_1} = (R_1 + R_2) I_1 - R_3 I_2$$

$R_1 + R_2 \equiv$ Self resistance of mesh 1

$-R_3 \equiv$ mutual resistance between meshes 1 and 2

Applying KVL for mesh 2 :

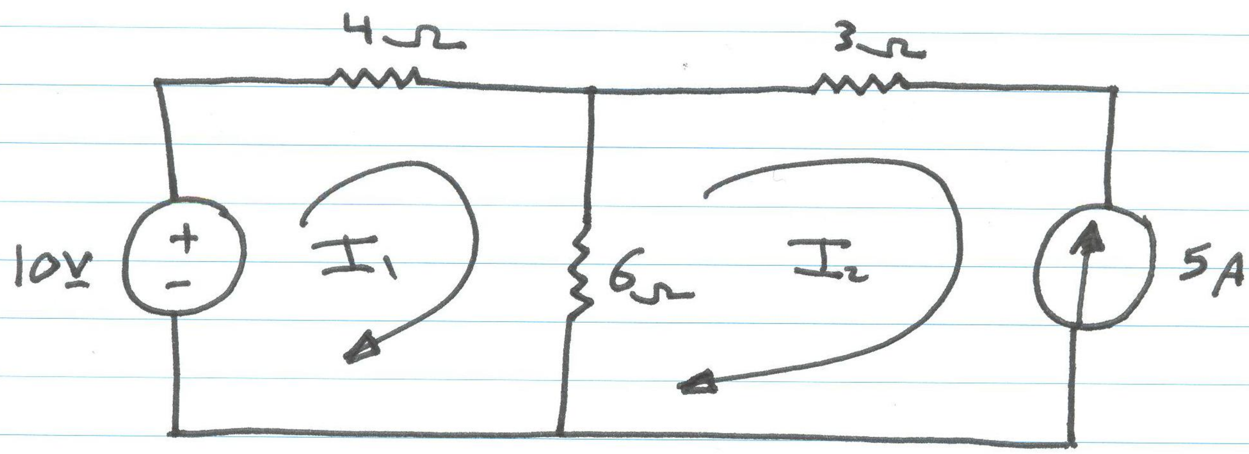
$$-V_{s_2} = -R_3 I_1 + (R_2 + R_3) I_2$$

$R_2 + R_3 \equiv$ Self resistance of mesh 2 .

Mesh Analysis : With Current Source

Care 1

Current source exist only in one mesh



KVL for mesh ①:

$$10 = 10 I_1 - 6 I_2$$

Constrain equation :

$$I_2 = -5A$$

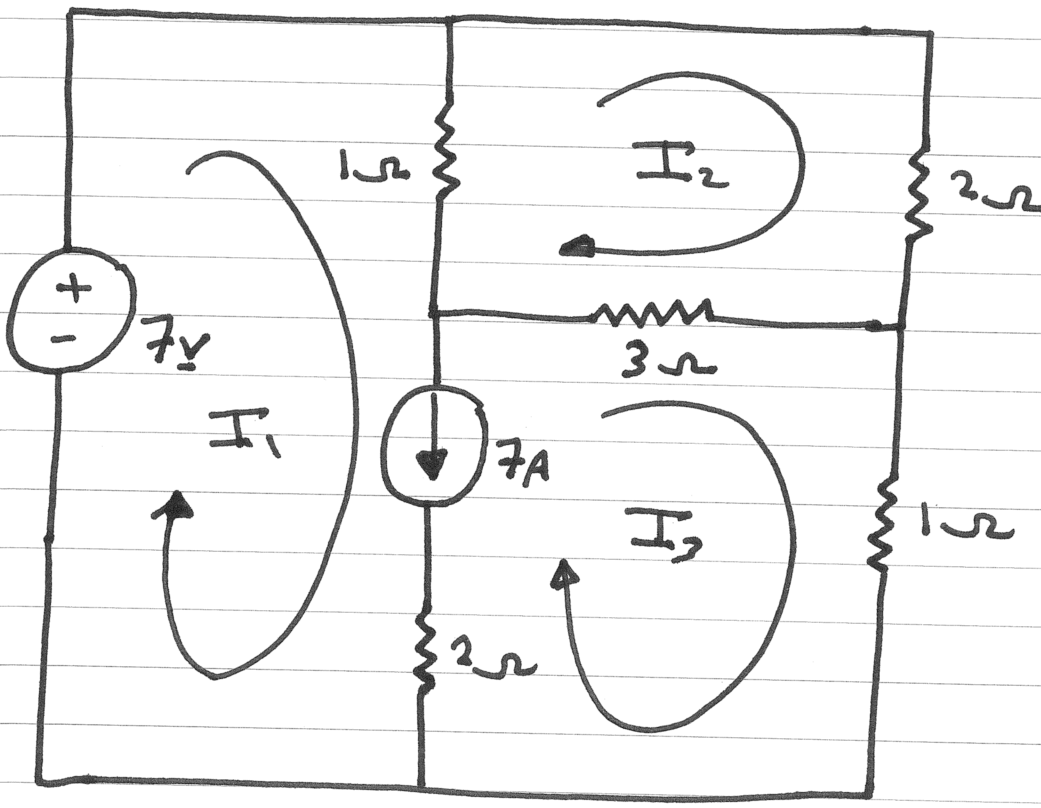
$$\therefore I_1 = -2A$$

Care 2

Current Source exists between two meshes,

a Super mesh is obtained

Mesh Analysis : With Current Sources



KVL for mesh ②:

$$0 = 6I_2 - I_1 - 3I_3$$

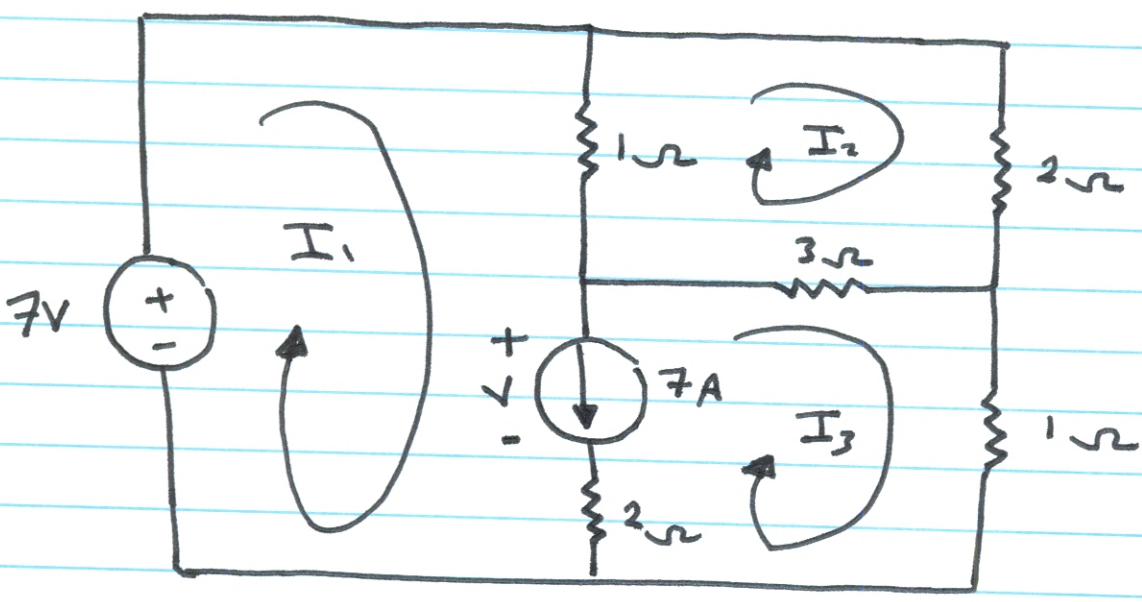
Constrain equation :

$$I_1 - I_3 = 7$$

Supermesh equation :

$$7 = I_1 + 4I_3 - 4I_2$$

Supermesh equation



KVL for mesh ① :

$$-7 + 1(I_1 - I_2) + V + 2(I_1 - I_3) = 0$$

$$7 = 3I_1 - I_2 - 2I_3 + V \quad \text{——} \quad ①$$

KVL for mesh ③ :

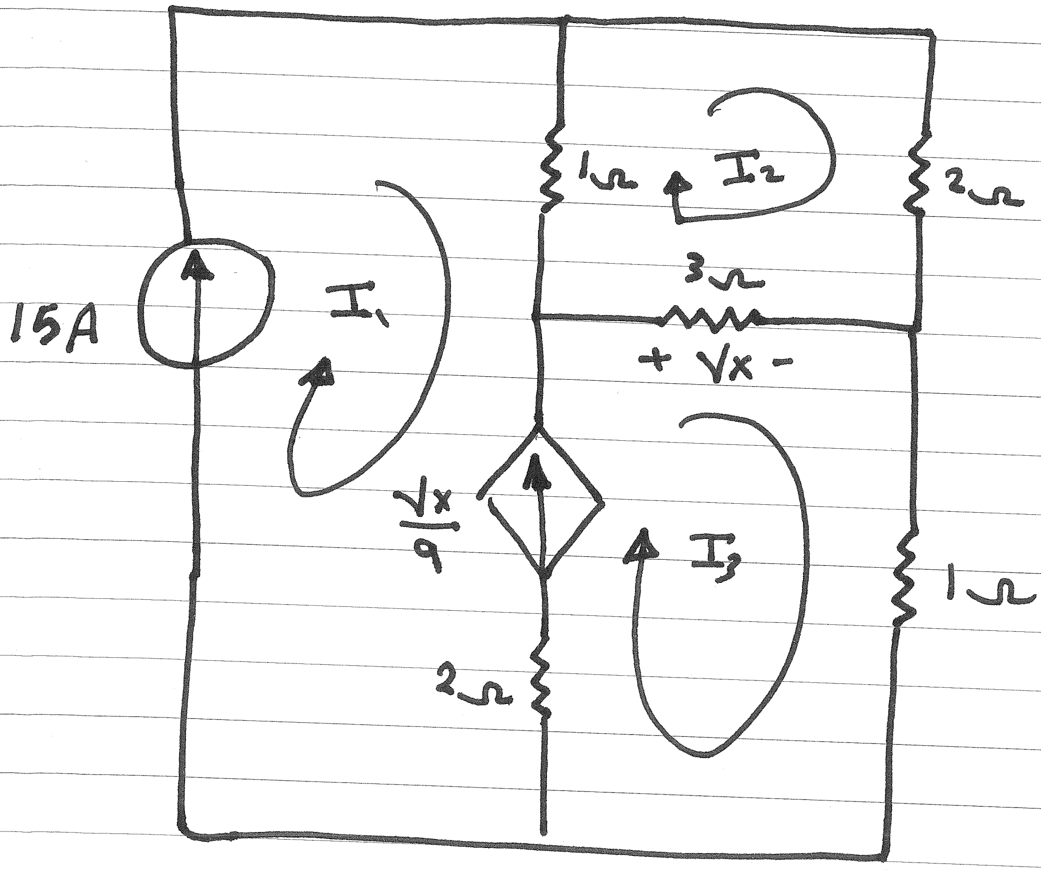
$$3(I_3 - I_2) + I_3 + 2(I_3 - I_1) - V = 0$$

$$0 = -2I_1 - 3I_2 + 6I_3 - V \quad \text{——} \quad ②$$

adding ① + ②

$$7 = I_1 - 4I_2 + 4I_3$$

Mesh Analysis : With dependent sources



KVL for mesh ②:

$$0 = -I_1 + 6I_2 - 3I_3$$

Constrain equation:

$$I_1 = 15A$$

Constrain equation:

$$I_3 - I_1 = \frac{V_x}{9}$$

$$V_x = 3\,(I_3 - I_2)$$

$\therefore \quad I_1 = 15A$

$I_2 = 11A$

$I_3 = 17A$

Node or Mesh : How to choose ?

- Use the one with fewer equations
- Use the method you Like best

Nodal and Mesh Analysis

- As Circuits get more Complicated, we need an organized method of applying KVL, KCL, and Ohm's
- Nodal analysis assigns voltages to each node then we apply KCL
- Mesh analysis assigns Currents to each mesh, and then we apply KVL

Nodal Analysis

Steps to Determine **Node Voltages**:

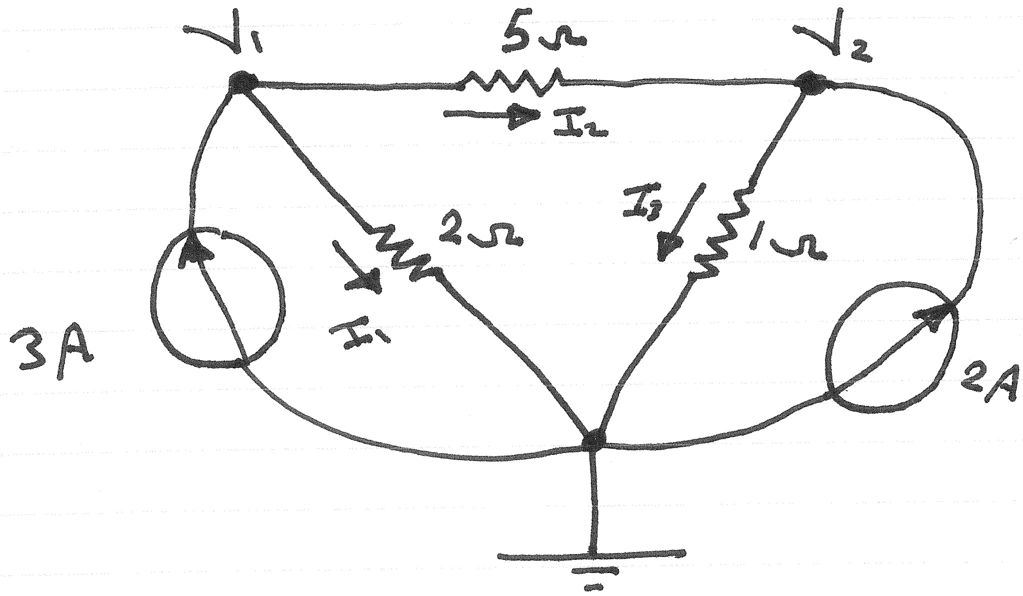
1. Select a node as the **reference node**. Assign voltage $\boldsymbol{V1, V2, \ldots V_{n-1}}$ to the remaining n-1 nodes. The voltages are referenced with respect to the reference node.

2. Apply **KCL** to each of the $\boldsymbol{n}$**-1 nonreference nodes**. Use Ohm’s law to express the branch currents in terms of node voltages.

3. Solve the resulting simultaneous equations to obtain the unknown node voltages.

The Nodal Analysis Method

assign voltages to every node relative to a reference node.



Apply KCL to node ①

$$3 = I_1 + I_2$$

$$3 = \frac{V_1}{2} + \frac{V_1 - V_2}{5}$$

$$3 = 0.7\,V_1 - 0.2\,V_2 \qquad \text{—— ①}$$

Apply KCL to node ②

$$2 + I_2 = I_3$$

$$2 = I_3 - I_2$$

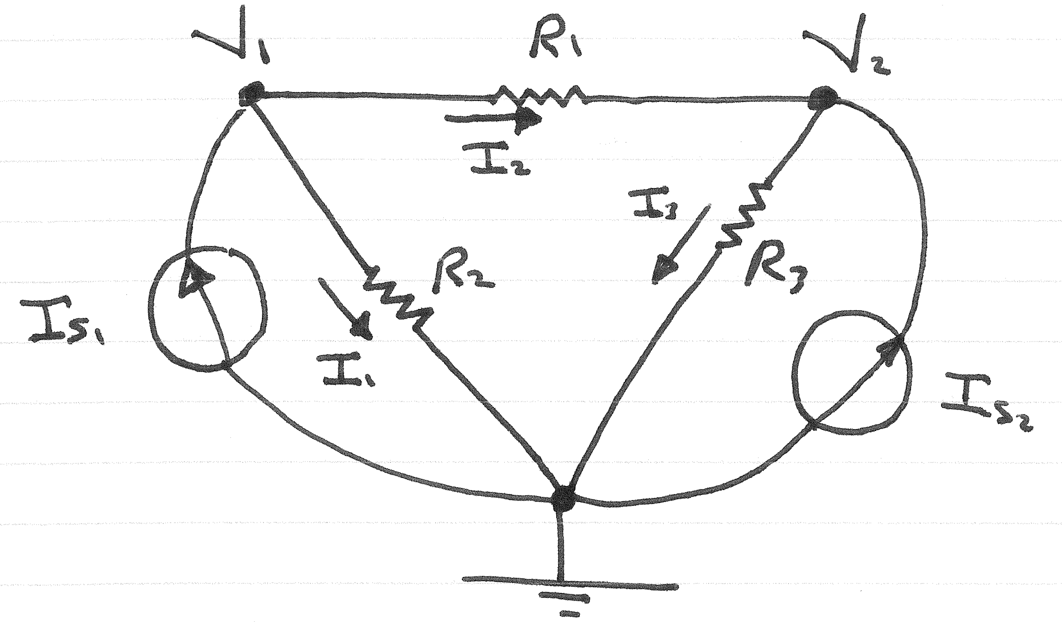
$$2 = \frac{V_2}{1} - \frac{V_1 - V_2}{5}$$

$$2 = -0.2V_1 + 1.2V_2 \quad \text{——————} \quad (2)$$

Solving equations (1) and (2), we get

$$V_1 = 5V$$

$$V_2 = 2.5V$$



Applying KCL to node 1

$$I_{S_1} = I_1 + I_2$$

$$I_{S_1} = \frac{V_1}{R_2} + \frac{V_1 - V_2}{R_1}$$

$$I_{S_1} = \left(\frac{1}{R_2} + \frac{1}{R_1}\right) V_1 - \frac{1}{R_1} V_2$$

$$I_{S_1} = (G_2 + G_1) V_1 - G_1 V_2$$

Self Conductance $= G_2 + G_1$

mutual Conductance $= -G_1$

Applying KCL to node ②

$$I_{s2} + I_2 = I_3$$

$$I_{s2} = I_3 - I_2$$

$$I_{s2} = \frac{V_2}{R_3} - \frac{V_1 - V_2}{R_1}$$

$$I_{s2} = -\frac{1}{R_1} V_1 + \left(\frac{1}{R_3} + \frac{1}{R_1} \right) V_2$$

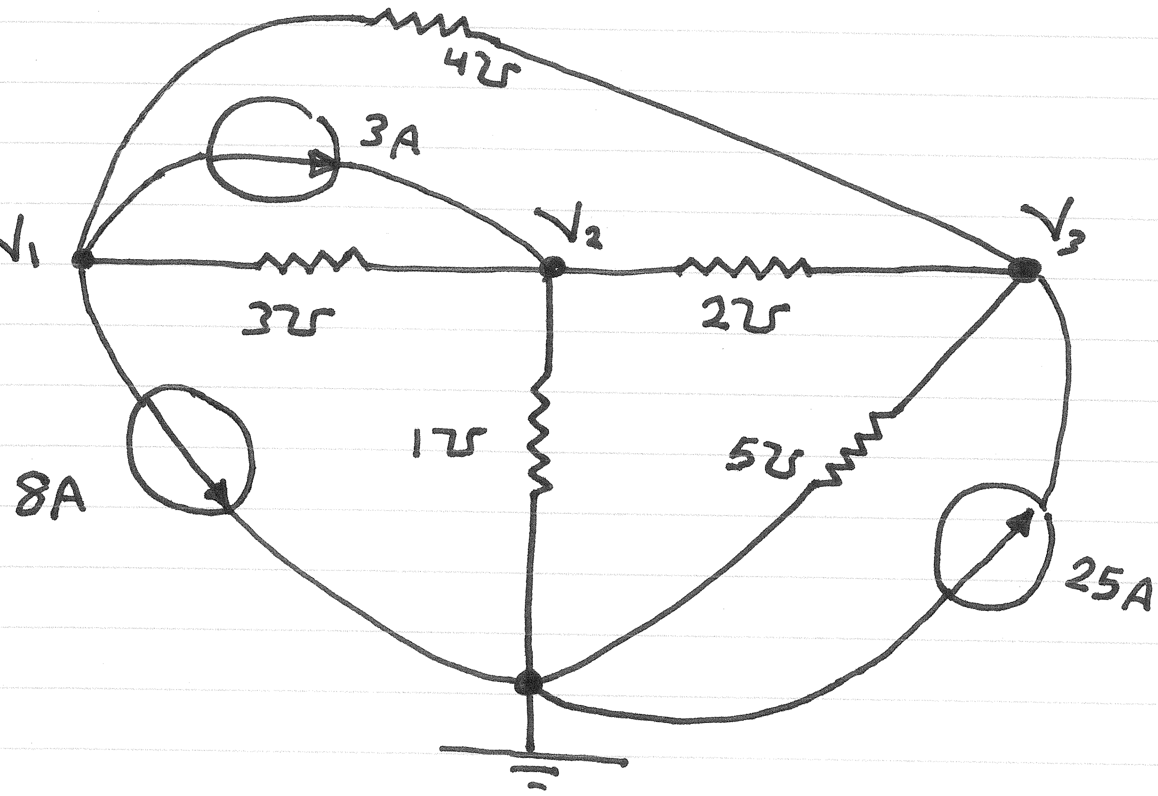
$$I_{s2} = -G_1 V_1 + (G_3 + G_1) V_2$$

Self Conductance of node ② $= (G_3 + G_1)$

mutual Conductance between nodes ① and ②

$= -G_1$

Writing Nodal equations by inspection



KCL at node ① :

$$7V_1 - 3V_2 - 4V_3 = -11$$

KCL at node ② :

$$-3V_1 + 6V_2 - 2V_3 = 3$$

KCL at node ③ :

$$-4V_1 - 2V_2 + 11V_3 = 25$$

Solving :

$V_1 = 1\,\underline{V}$; $V_2 = 2\,\underline{V}$; $V_3 = 3\,\underline{V}$

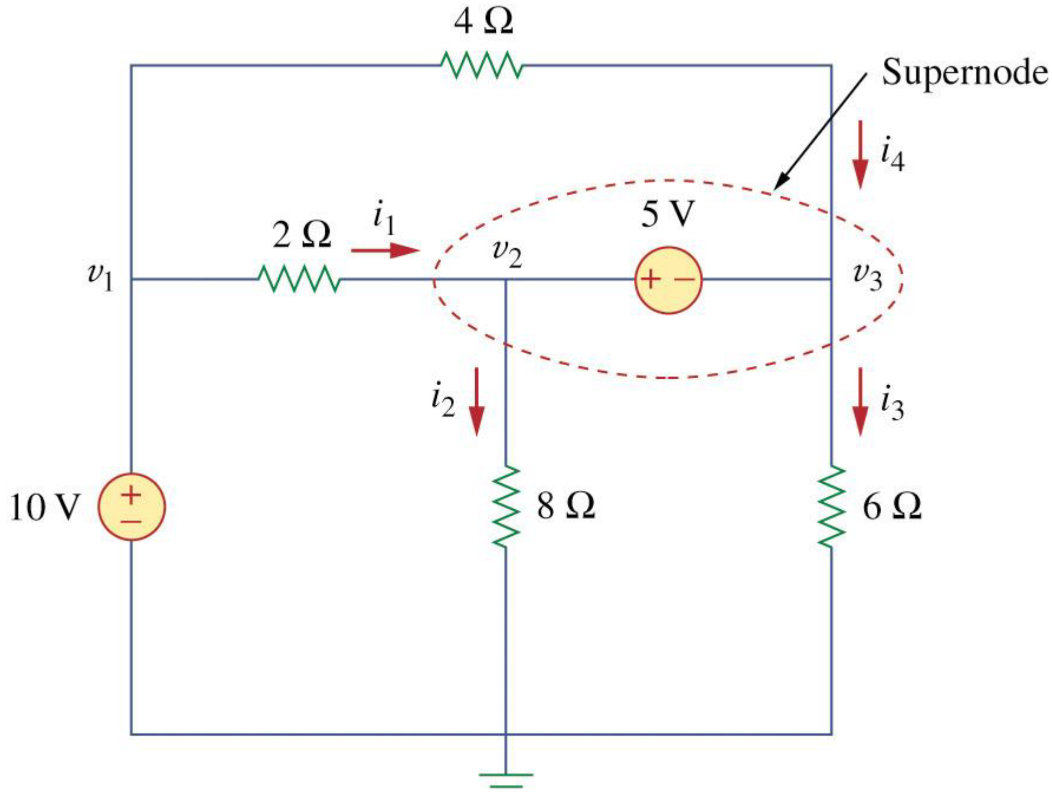
Nodal Analysis with Voltage Sources

Case 1: The voltage source is connected between a non reference node and the reference node: The non reference node voltage is equal to the magnitude of voltage source and the number of unknown non reference nodes is reduced by one.

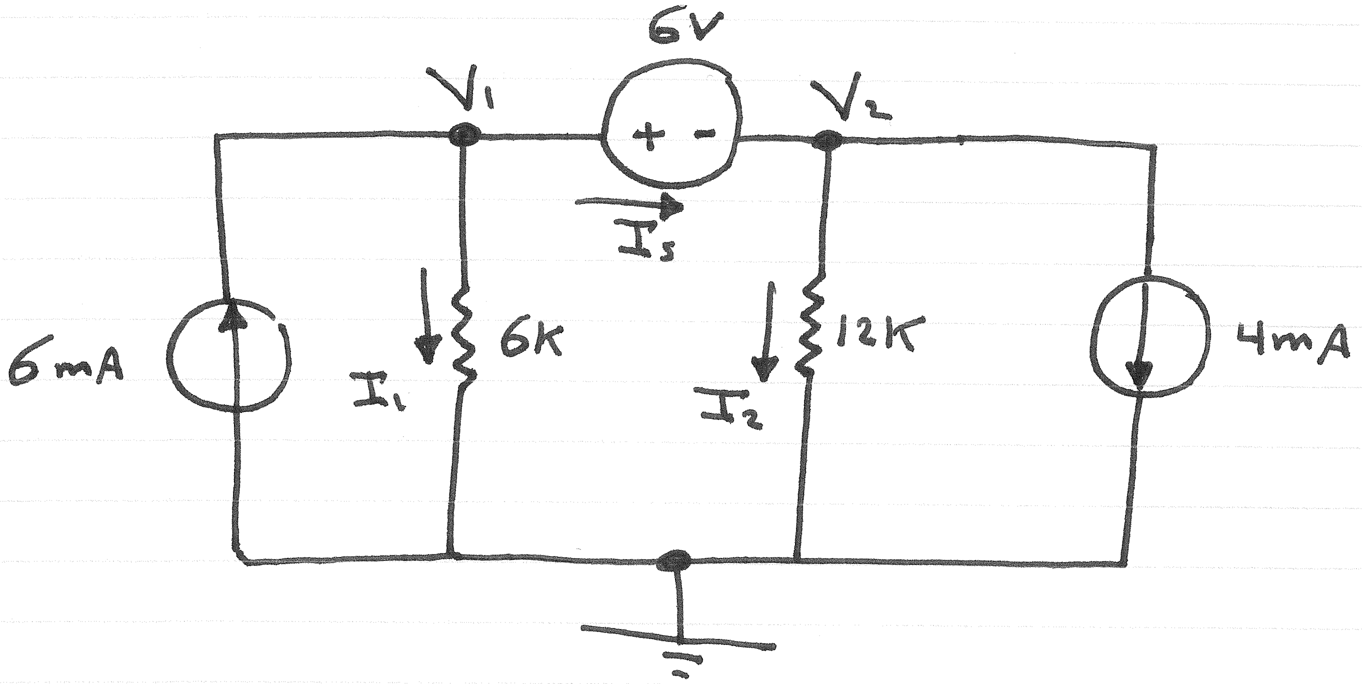
Case 2: The voltage source is connected between two non referenced nodes: a generalized node (**supernode**) is formed.

Nodal Analysis with Voltage Sources

A circuit with a **supernode**.



Voltage Sources and the supernode



Constrain equation :

$$V_1 - V_2 = 6 \qquad \text{———} \quad (1)$$

KCL at node (1) :

$$6mA = I_1 + I_s$$

$$6mA = \frac{V_1}{6K} + I_s \qquad \text{———} \quad (2)$$

KCL at node (2) :

$$I_s = I_2 + 4mA$$

$$4mA = I_s - I_2$$

$$4\,mA = I_S - \frac{V_2}{12K} \quad \text{———} \quad (3)$$

Subtracting (3) from (2)

$$2\,mA = \frac{V_1}{6K} + \frac{V_2}{12K} \quad \text{———} \quad (4)$$

This is the supernode equation

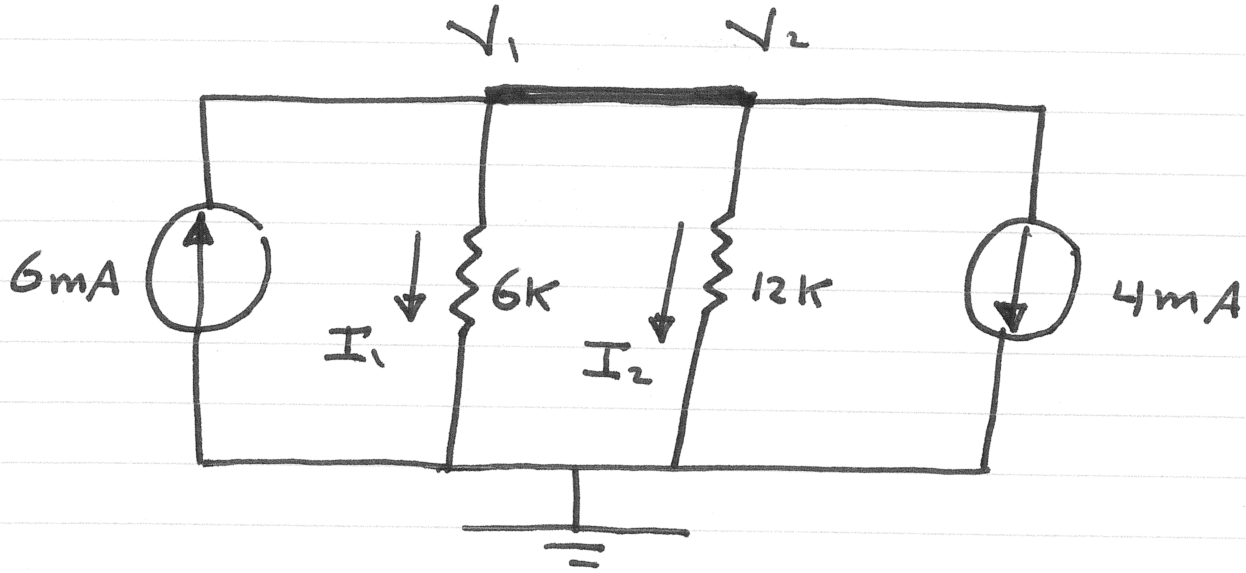
Solving (1) and (4)

we get

$$V_1 = 10\,V$$

$$V_2 = 4\,V$$

Supernode equation by inspection

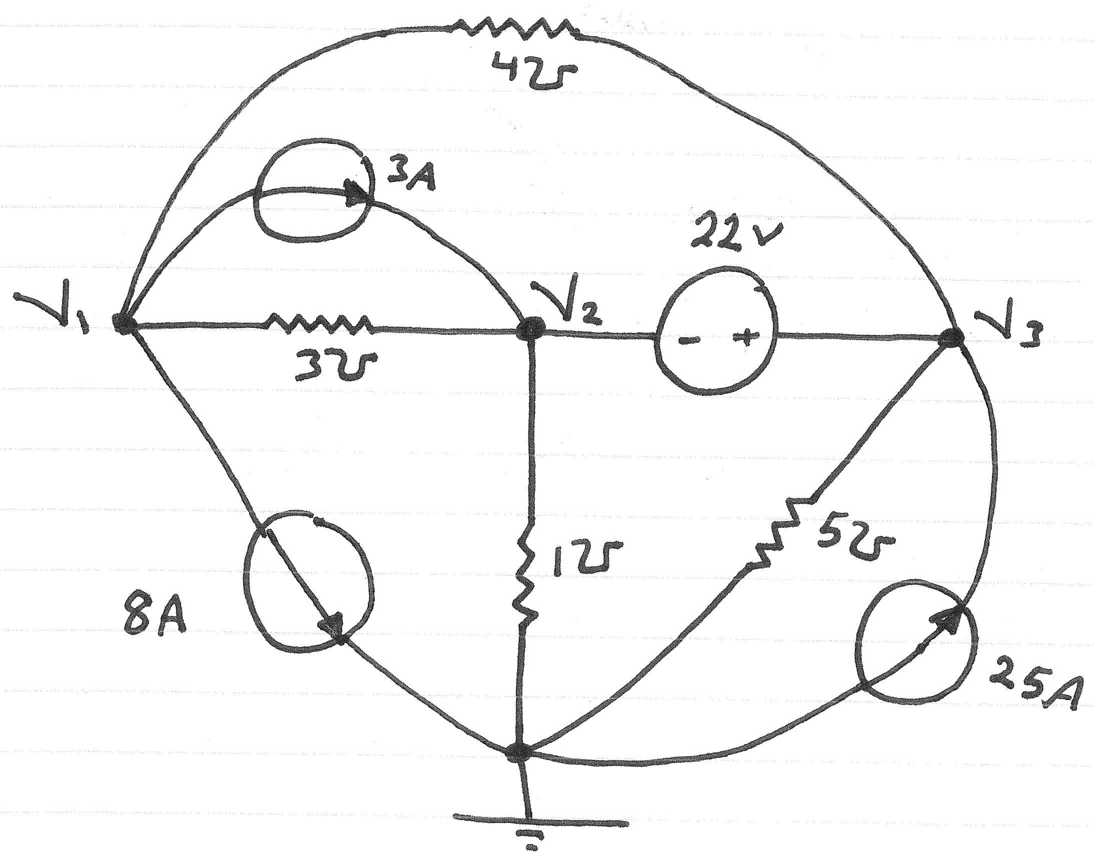


$$6\,mA = I_1 + I_2 + 4\,mA$$

$$2\,mA = I_1 + I_2$$

$$2\,mA = \frac{V_1}{6K} + \frac{V_2}{12K}$$

Voltage Sources and the Supernode



Constrain equation :

$$V_3 - V_2 = 22$$

KCL at node ① :

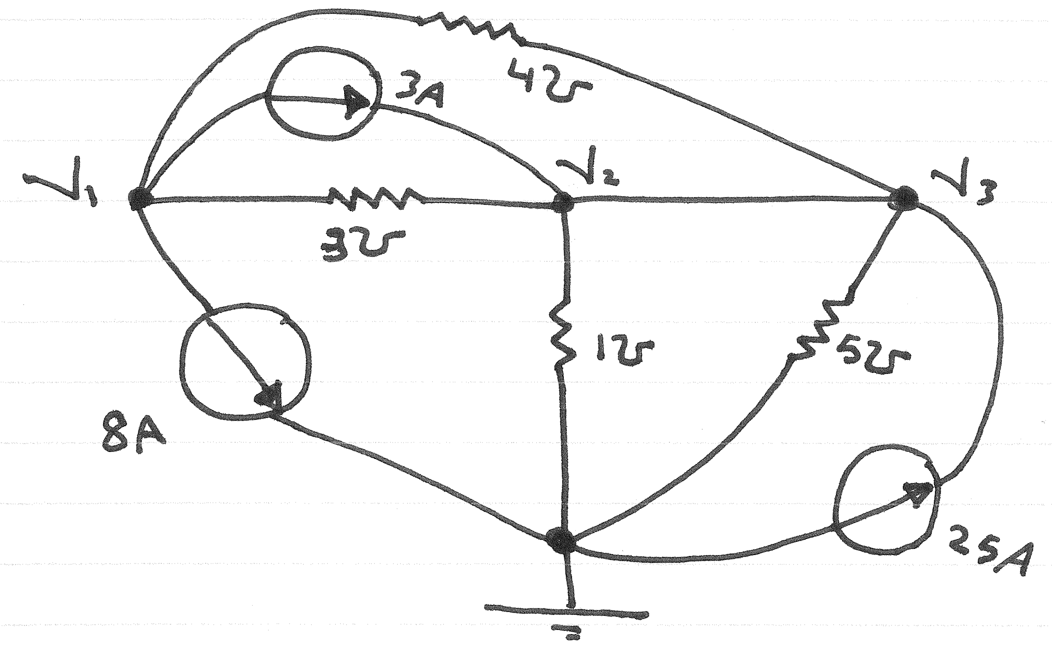
$$7V_1 - 3V_2 - 4V_3 = -11$$

Supernode equation :

$$-7V_1 + 4V_2 + 9V_3 = 28$$

Solving for $V_1 \Rightarrow V_1 = -4.5\,\underline{V}$

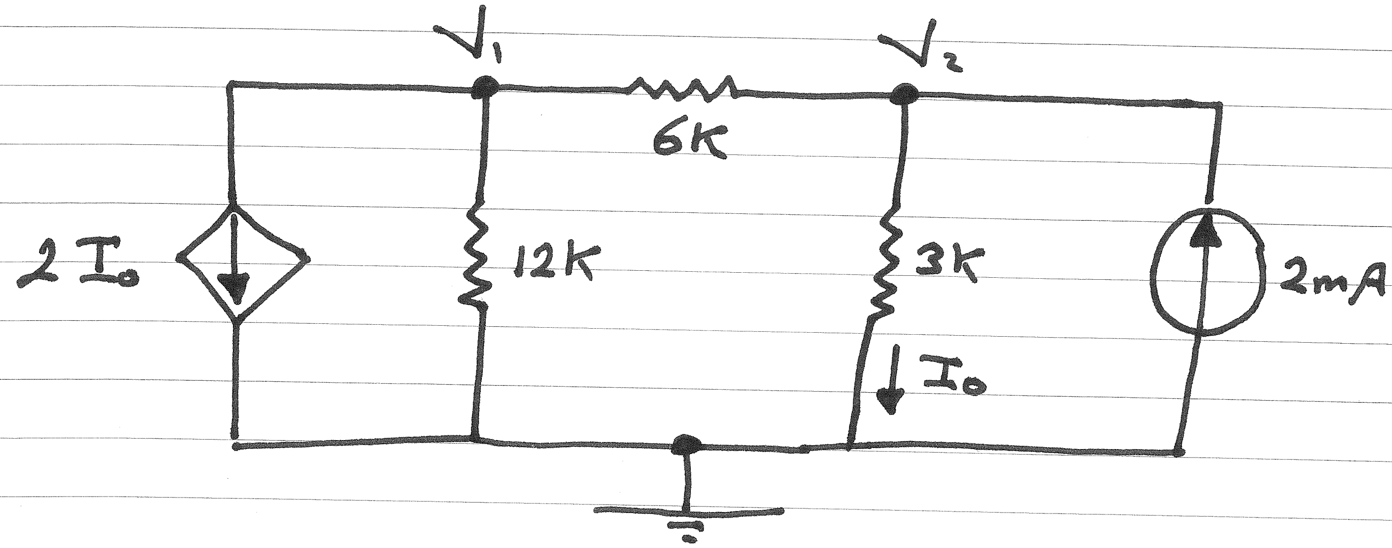
Supernode equation by inspection



$$25 + 3 = (1+3)V_2 + (5+4)V_3 - (3+4)V_1$$

$$28 = 4V_2 + 9V_3 - 7V_1$$

Circuits with dependent sources



KCL at node 1 :

$$-2I_o = \left(\frac{1}{12K} + \frac{1}{6K}\right)V_1 - \frac{1}{6K}V_2$$

$$I_o = \frac{V_2}{3K}$$

$$0 = \left(\frac{1}{12K} + \frac{1}{6K}\right)V_1 + \left(\frac{2}{3K} - \frac{1}{6K}\right)V_2$$

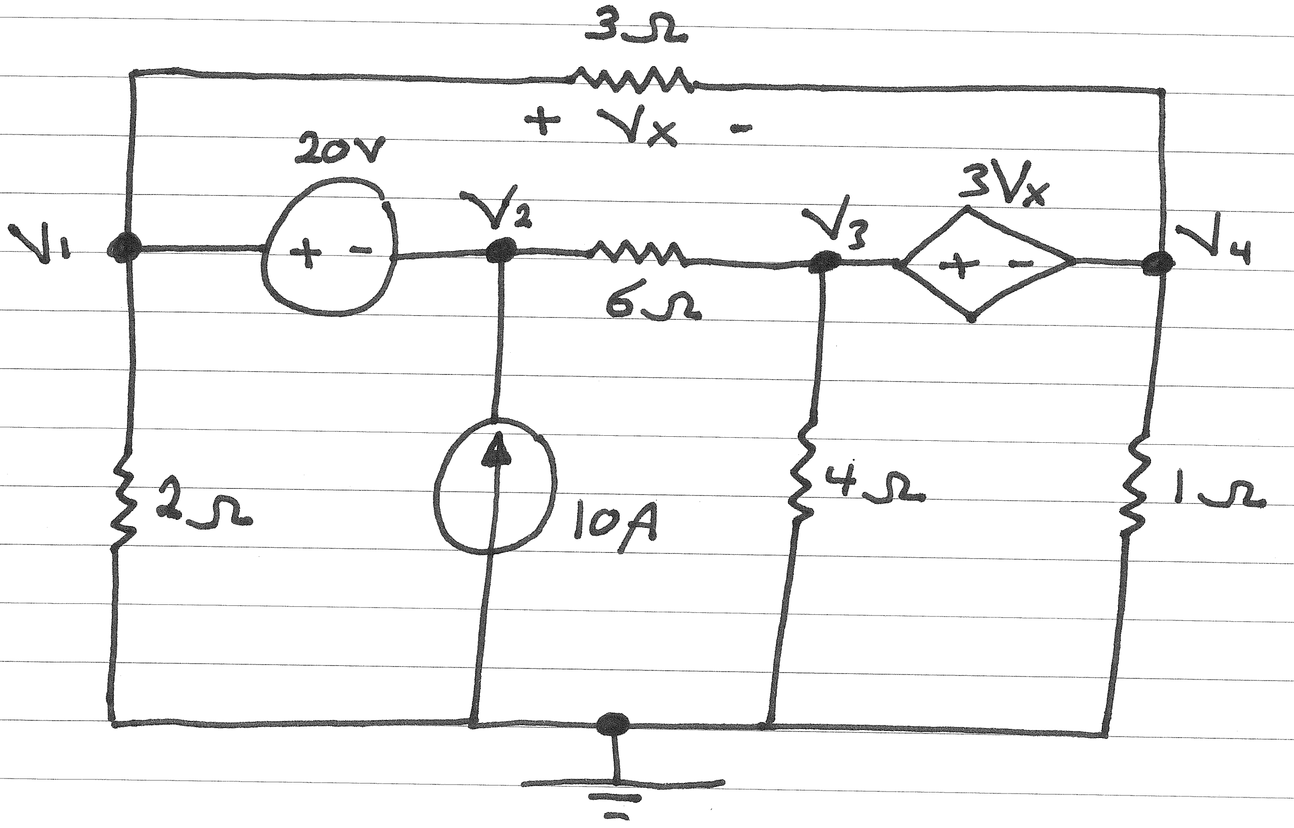
KCL at node 2 :

$$2mA = -\frac{1}{6K}V_1 + \left(\frac{1}{3K} + \frac{1}{6K}\right)V_2$$

Solving for V_1 and V_2 , we get

$$V_1 = -\frac{24}{5}\,\underline{V} \quad ; \quad V_2 = \frac{12}{5}\,\underline{V}$$

Nodal Analysis : Supernode



KCL at Supernode 1,2 :

$$10 = \left(\frac{1}{2}+\frac{1}{3}\right)V_1 + \frac{1}{6}V_2 - \frac{1}{6}V_3 - \frac{1}{3}V_4$$

KCL at Supernode 3,4 :

$$0 = \left(\frac{1}{4}+\frac{1}{6}\right)V_3 + \left(\frac{1}{1}+\frac{1}{3}\right)V_4 - \frac{1}{6}V_2 - \frac{1}{3}V_1$$

Constrain equation :

$$V_1 - V_2 = 20$$

Constrain equation :

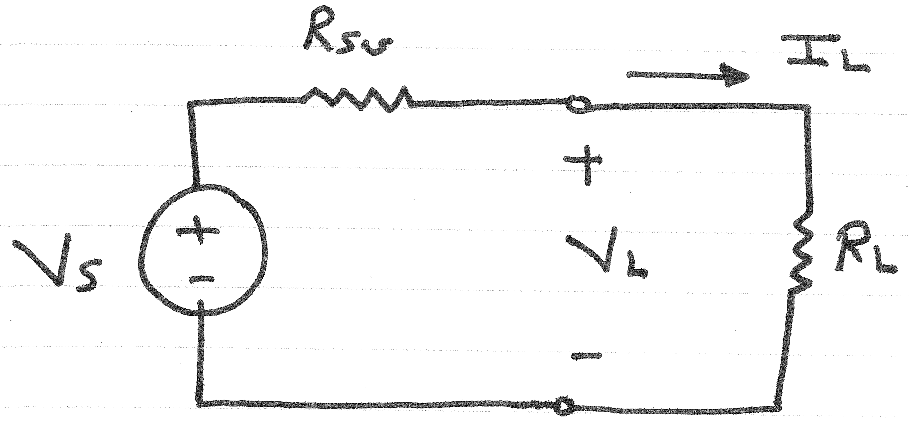
$V_3 - V_4 = 3V_x$

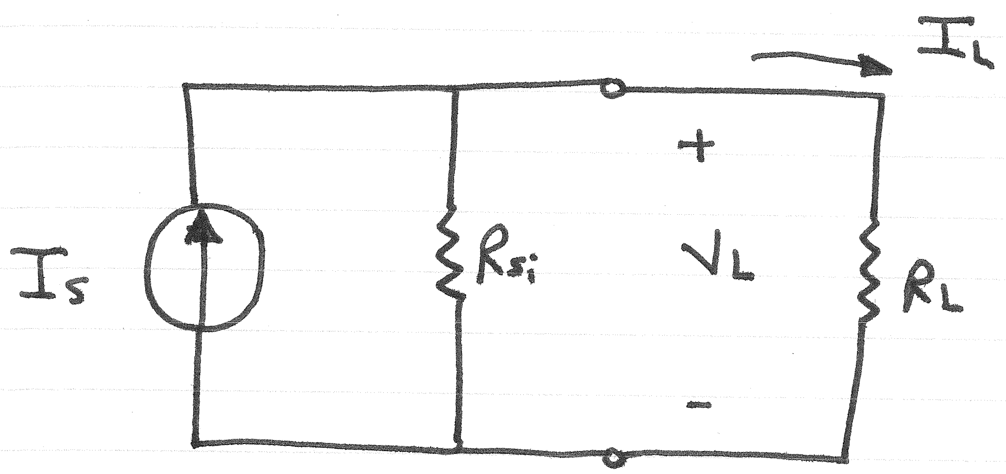
$V_x = V_1 - V_4$

$0 = 3V_1 - V_3 - 2V_4$

What is the current through the independent voltage source ?

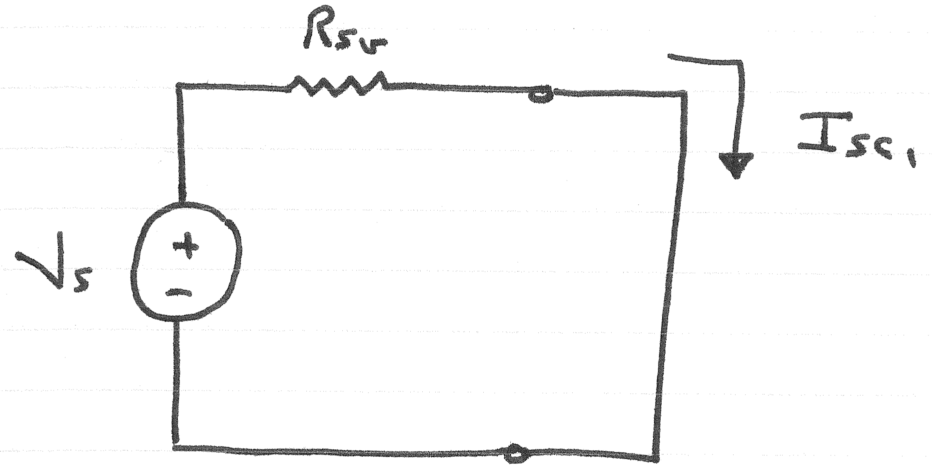
Source Transformation



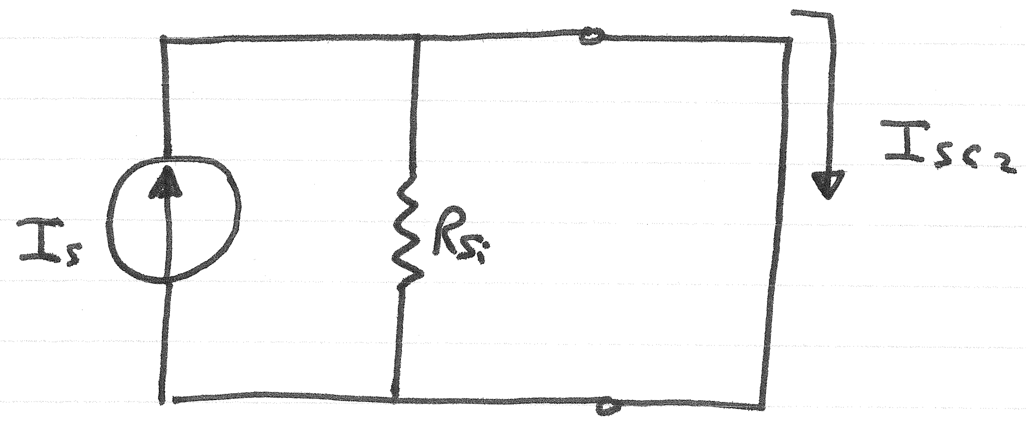


Two sources are equivalent, if each produces identical current and identical voltage in any Load which is placed across its terminal.

1) let $R_L = 0$ (short circuit)



$$I_{sc_1} = \frac{V_s}{R_{sv}}$$

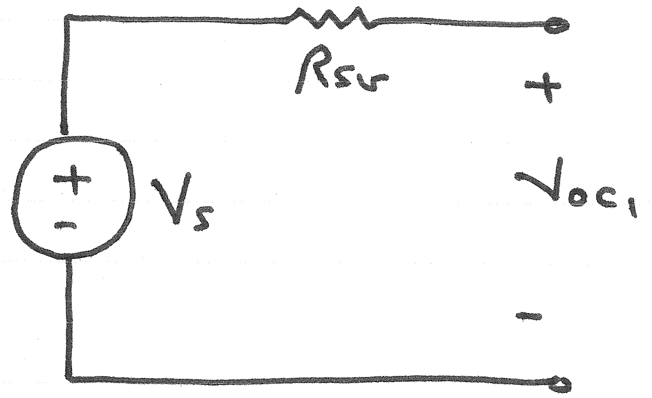


$$I_{sc_2} = I_s$$

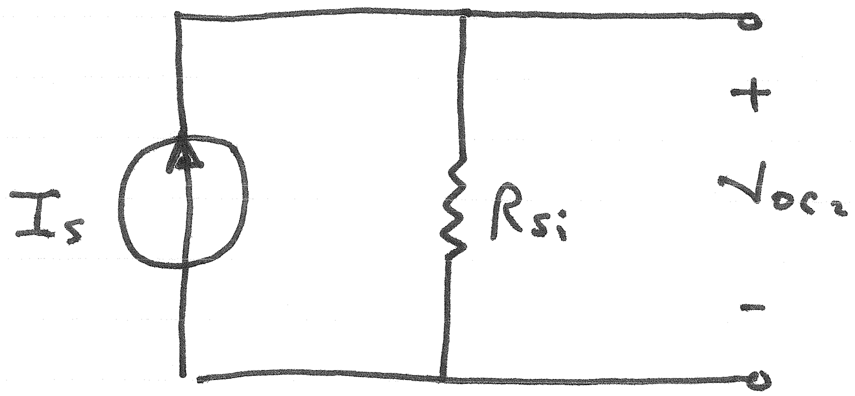
For $I_{sc_1} = I_{sc_2}$

$$\boxed{\frac{V_s}{R_{sv}} = I_s} \qquad \text{———} \quad (1)$$

2) Let $R_L = \infty$ (open circuit)



$$V_{oc1} = V_s$$



$$V_{oc2} = I_s R_{si}$$

For $V_{oc1} = V_{oc2}$

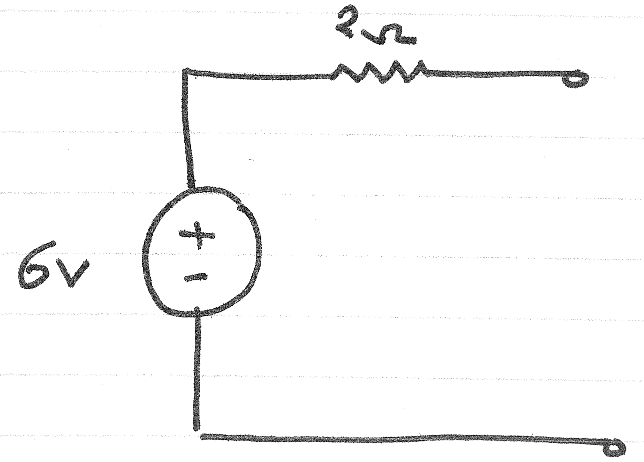
$$\boxed{V_s = I_s R_{si}} \quad \text{— (2)}$$

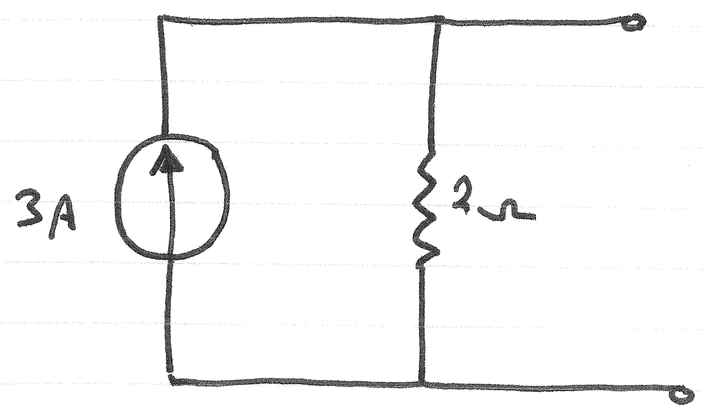
$$V_s = I_s R_{si}$$

using equation ①, we get

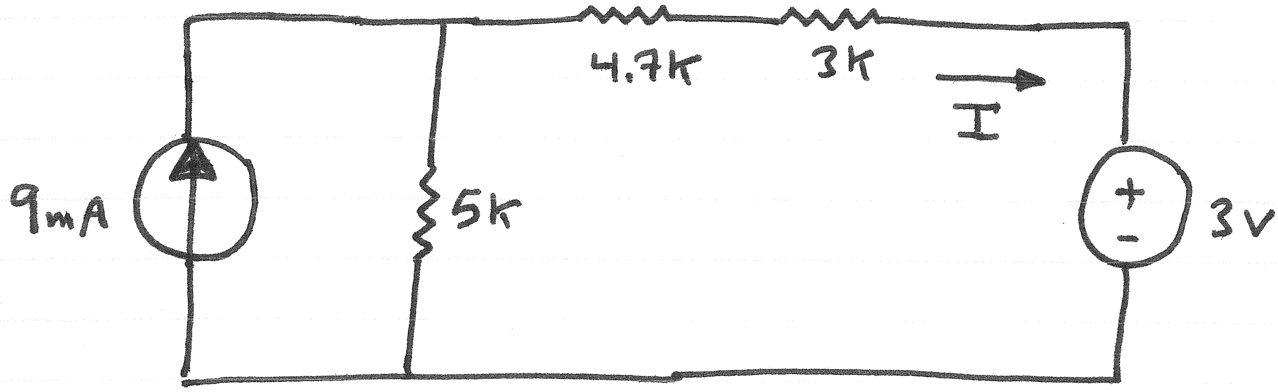
$$V_s = I_s R_{sv}$$

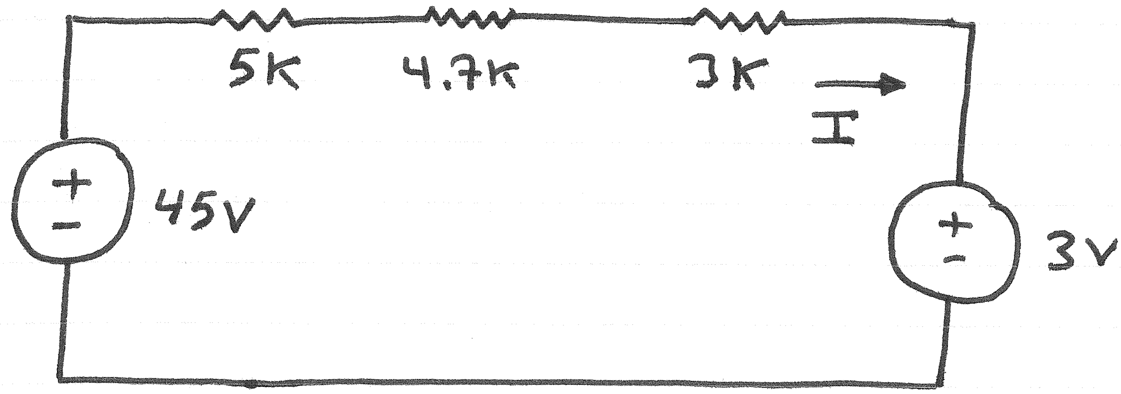
$$\therefore \quad R_{si} = R_{sv}$$





Find I using source transformation

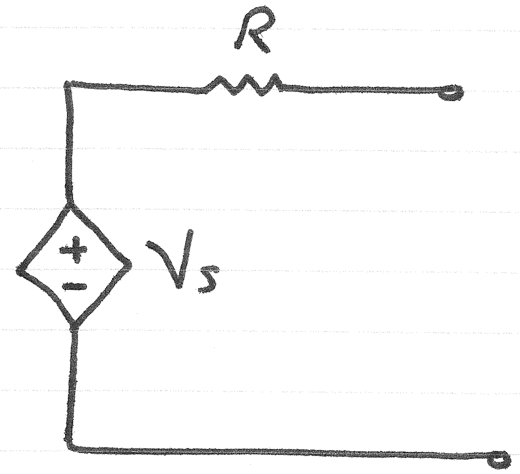


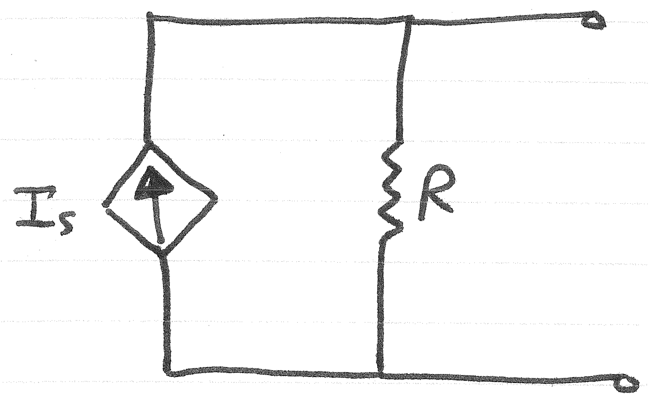


$$I = \frac{45-3}{5K+4.7K+3K} = 3.3\ mA$$

Source Transformation

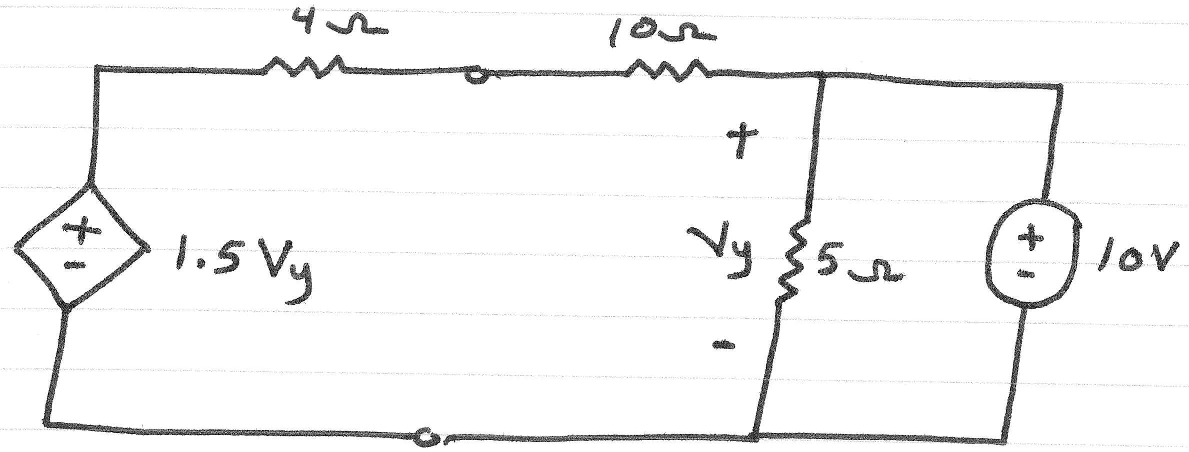
Dependent sources



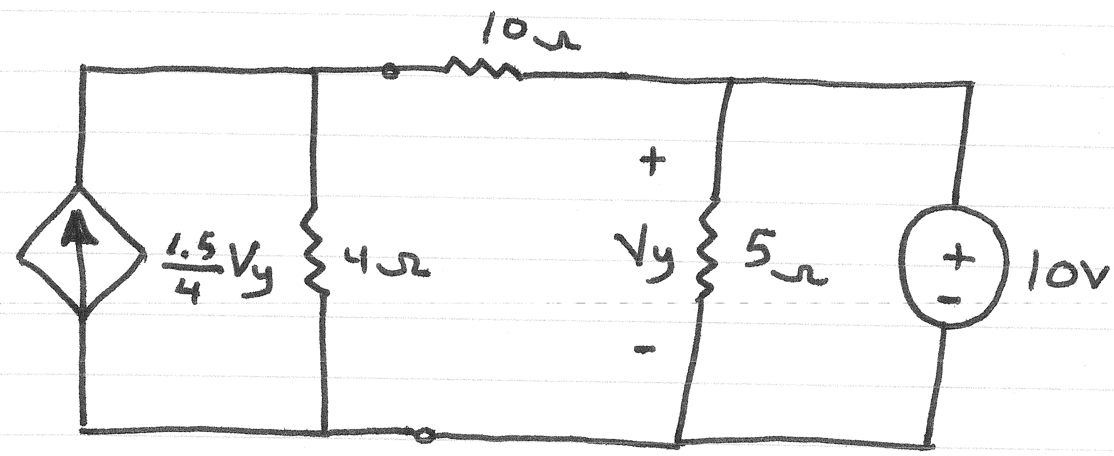


$$I_s = \frac{V_s}{R}$$

The Control variable must be outside the transformation.

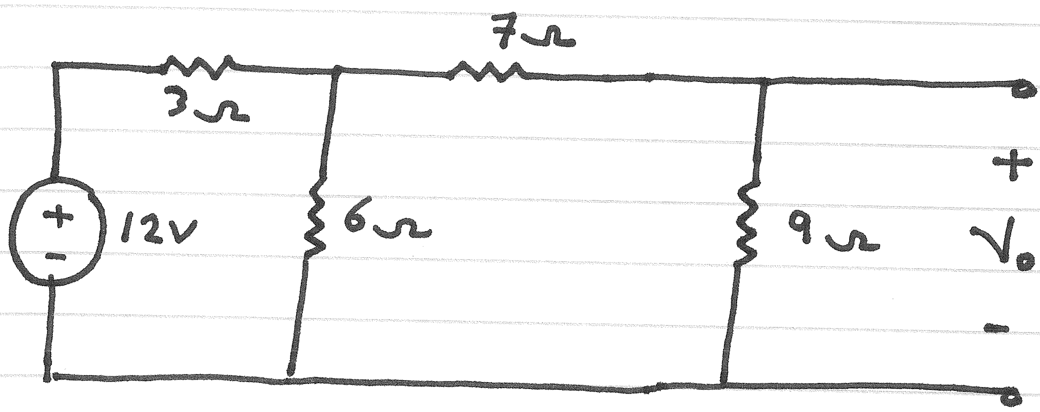


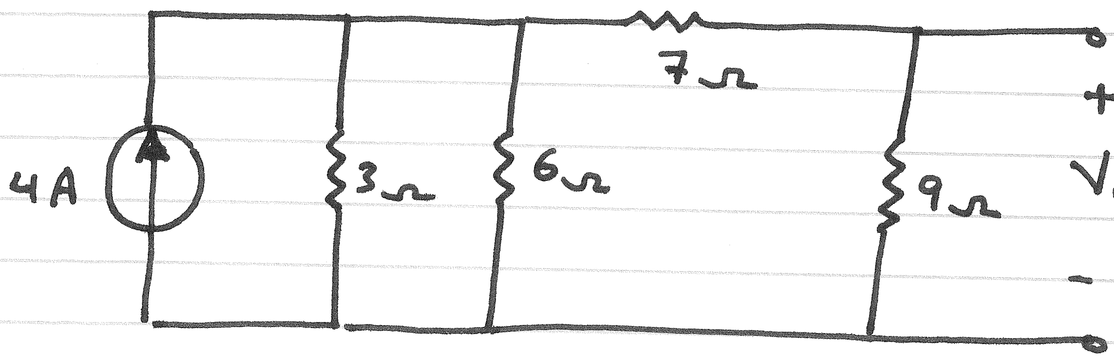
4Ω
10Ω
+
$1.5\,V_y$
V_y
5Ω
+
10V
-

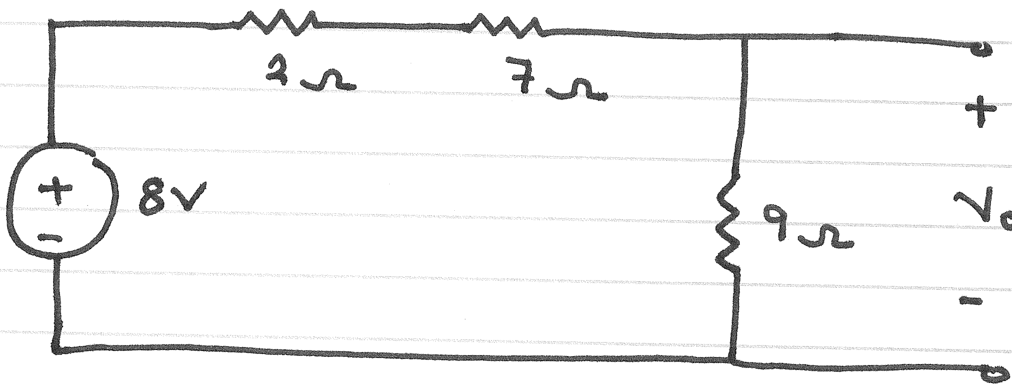


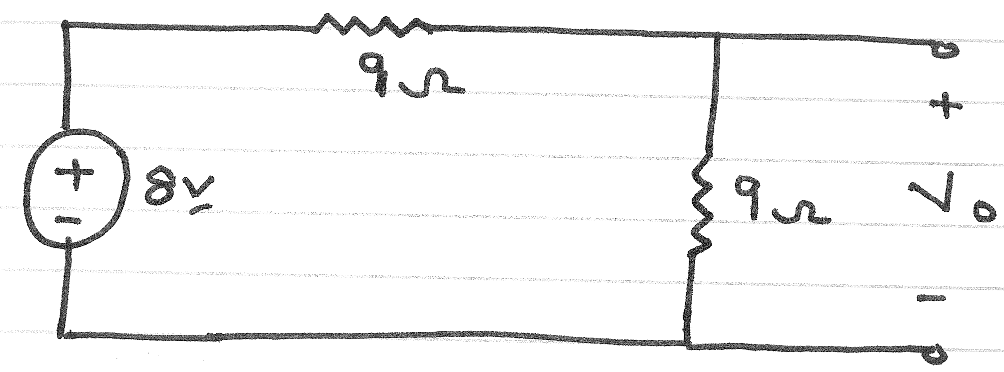
10Ω
$\frac{1.5}{4}V_y$
4Ω
+
V_y
5Ω
+
10V
-

Find V_o using source transformation









$$V_o = \frac{9}{9+9} \cdot 8V = 4V$$

The Superposition Theorem

In a Linear network, the voltage across or the current through any element may be calculated by adding algebraically all the individual voltages or currents caused by the seperate independent sources acting alone, i.e. with

1) all other independent voltage sources replaced by Short Circuits and
2) all other independent current Sources replaced by open circuits.

* Dependent Sources are left intact because they are Controlled by circuit variables.

Linear Elements and Circuits

- a Linear circuit element has a Linear voltage - current relationship

$$v(t) = R\,i(t)$$

$$v(t) = \frac{1}{C}\int_{-\infty}^{t} i(t)\,dt$$

$$v(t) = L\,\frac{di(t)}{dt}$$

- Independent sources are Linear elements

- Dependent sources need Linear control equation to be Linear elements

- Linear Circuit is a Circuit composed entirely of independent sources, Linear dependent sources, and Linear elements

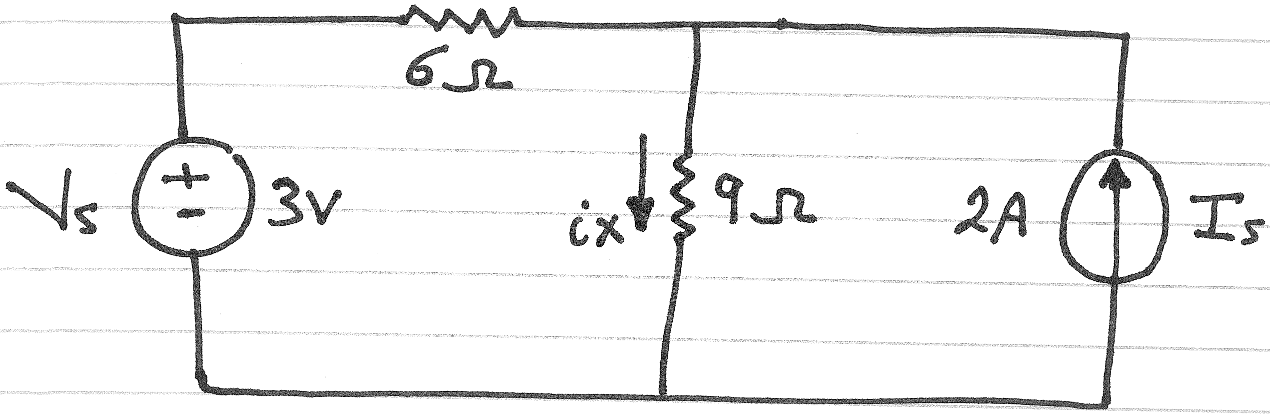
Steps to apply superposition principle

1. Turn off all indepedent sources except one source. Find the output (voltage or current) due to that source using nodal, mesh, ------ .

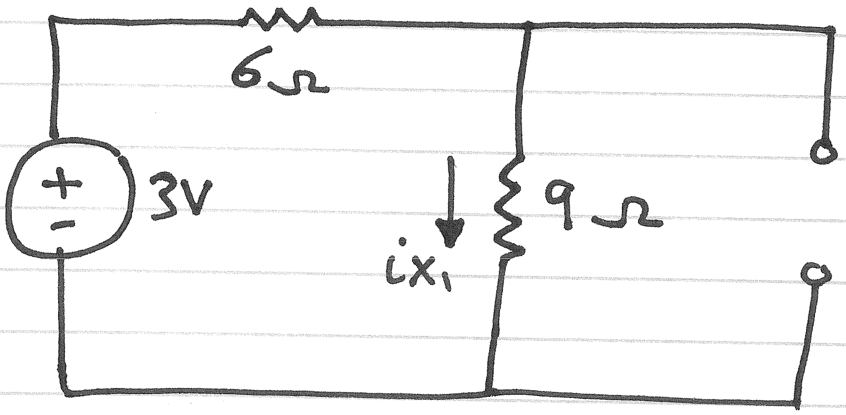
2. Repeat step 1 for each of the other independent sources.

3. Find the total Contribution by adding algebraically all the Contributions due to each independent sources.

Use superposition to solve for i_x

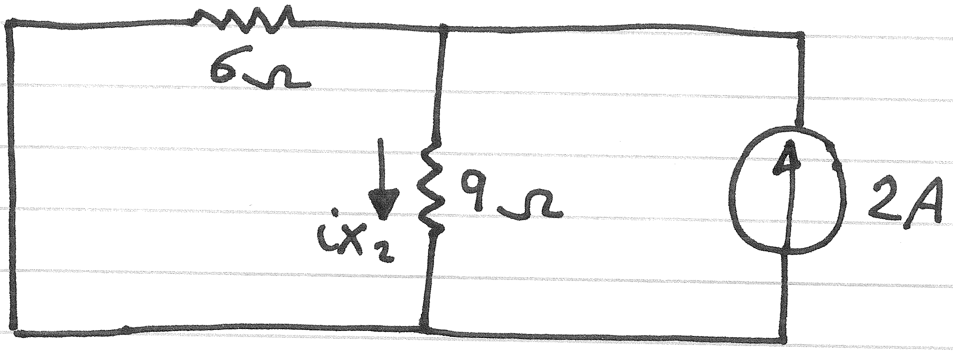


1) Let V_s on, and turn off I_s



$$i_{x_1} = \frac{3}{15} = 0.2\,A$$

2) let I_s on, and turn off V_s



$$i_{x_2} = \frac{6}{6+9}(2) = 0.8\,A$$

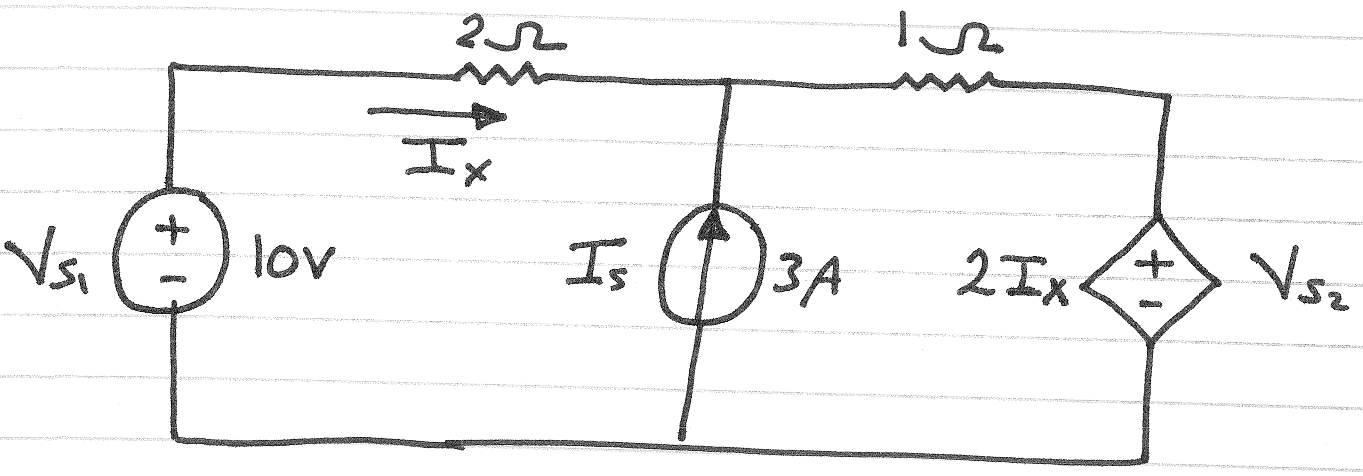
Finally, Combine the results :

$$i_x = i_{x_1} + i_{x_2}$$

$$= 0.2A + 0.8A$$

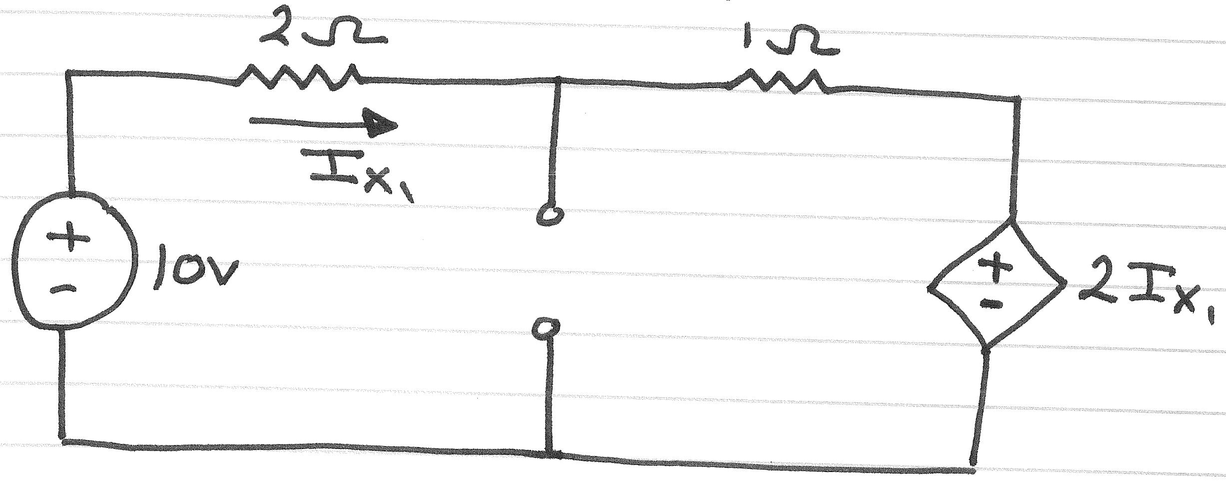
$$i_x = 1A$$

Superposition with a Dependent Source



Find I_x using superposition

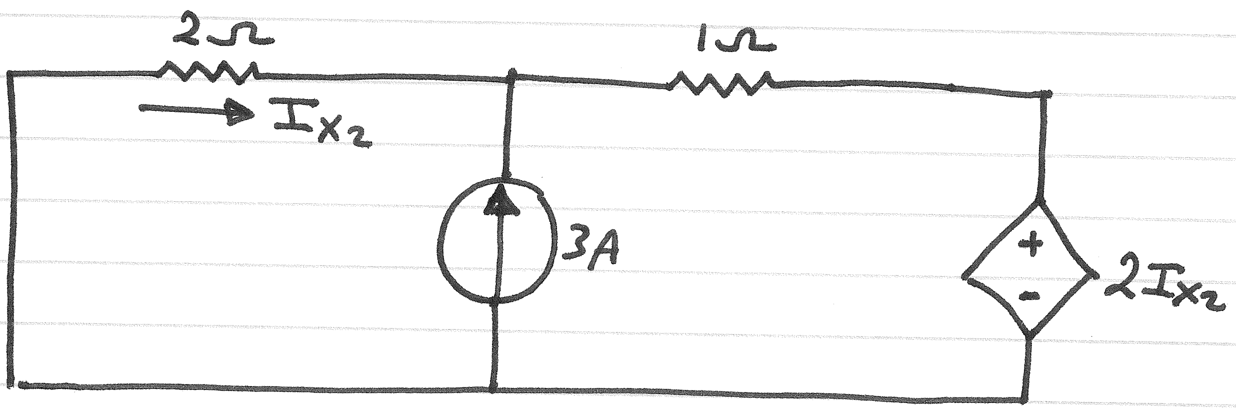
1) Let V_{S_1} on, and turn off I_s

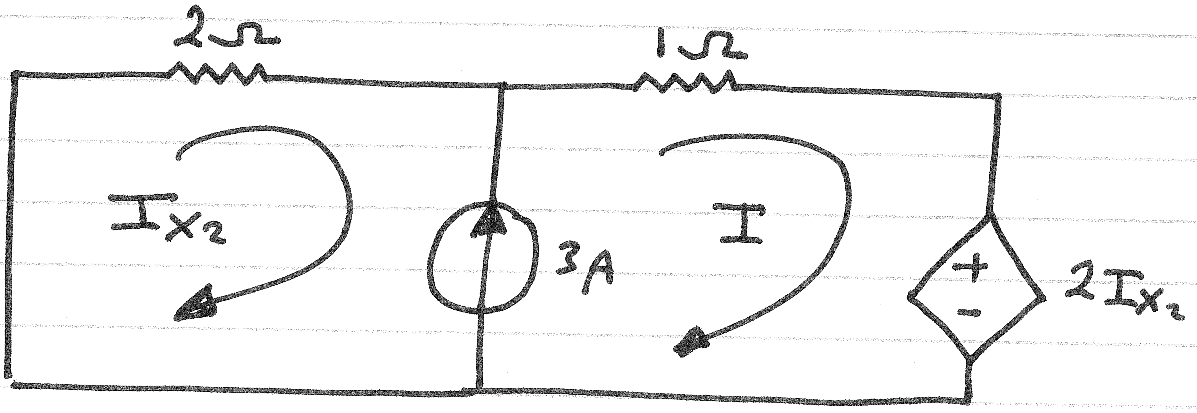


$$2I_{x_1} + I_{x_1} + 2I_{x_1} - 10 = 0$$

$$\therefore \quad I_{x_1} = 2A$$

2) let I_s on, and turn off V_{s1}





$3 = I - I_{x2}$ Constrain equation

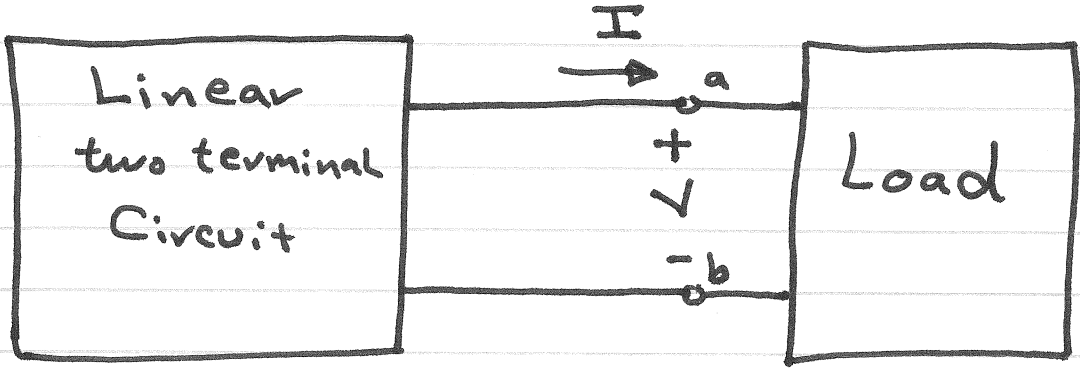
$-2I_{x2} = 2I_{x2} + I$ supermesh equation

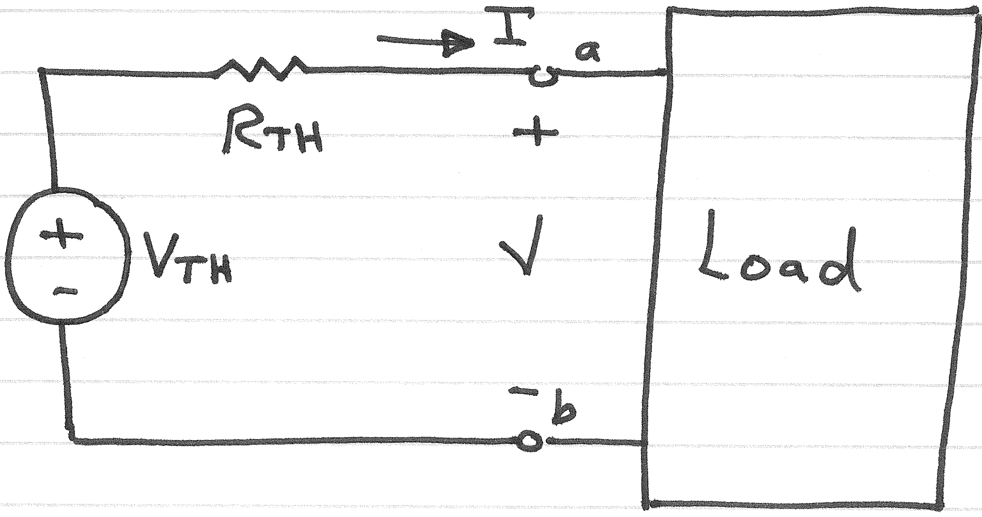
$\therefore I_{x2} = -0.6\,A$

$I_x = I_{x1} + I_{x2} = 2 - 0.6 = 1.4\,A$

* when applying superposition to circuits with dependent sources, these dependent Sources are never turned off.

Thevenin's Theorem

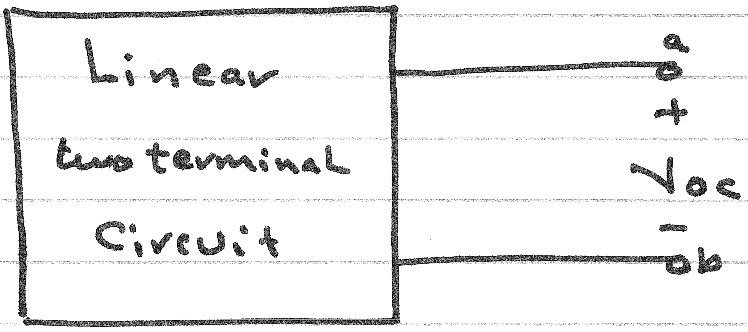




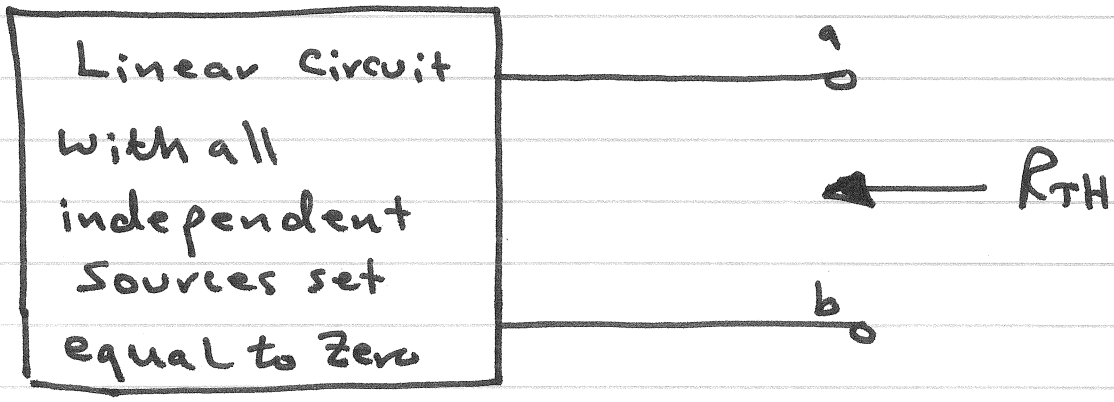
It states that a Linear two terminal Circuit Can be replaced by an equivalent Circuit Consisting of a voltage Source V_{TH} in series with a resistor R_{TH}, where V_{TH} is the open circuit voltage at the terminals and R_{TH} is the input or equivalent resistance at the terminals when the independent sources are turn off.

How to find Thevenin's Voltage?

$V_{TH} = V_{oc}$



How to find Thevenin's Resistance?



- a-b open circuited
- Turn off all independent sources

How to find R_{TH} ?

Case I

If the circuit has no dependent sources

- Turn off all independent sources
- R_{TH} Can be obtained via simplification of either parallel or series Connection seen from a-b.

Case II

If the circuit has dependent sources

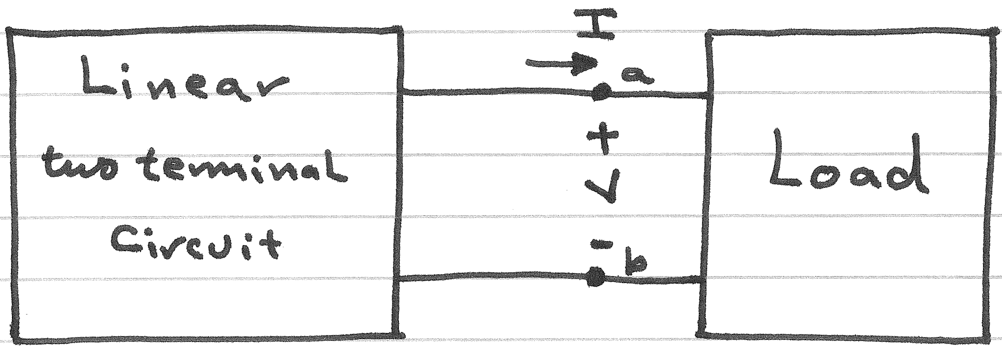
- Turn off all independent sources
- Apply a voltage source V_T at a-b

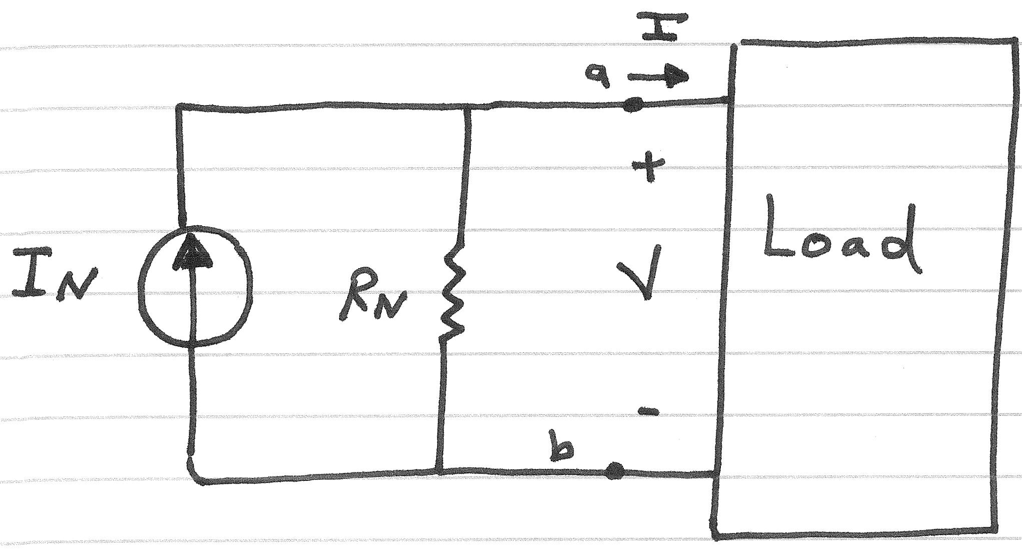
$$R_{TH} = \frac{V_T}{I_T}$$

- Alternatively, Apply a current source I_T at a-b

$$R_{TH} = \frac{V_T}{I_T}$$

Norton's Theorem





It states that a Linear two terminal Circuit Can be replaced by an equivalent Circuit of a Current source I_N in parallel With a resistor R_N.

where

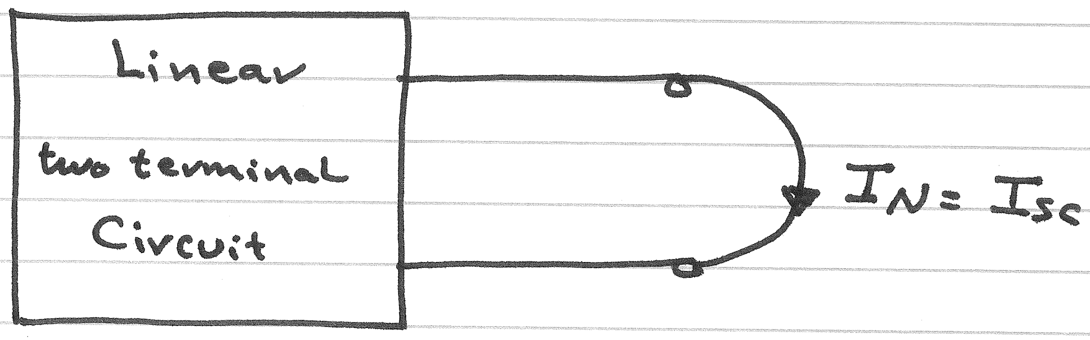
- I_N is the short Circuit current through the terminals.

- R_N is the input or equivalent resistance at the terminals when the independent sources are turned off.

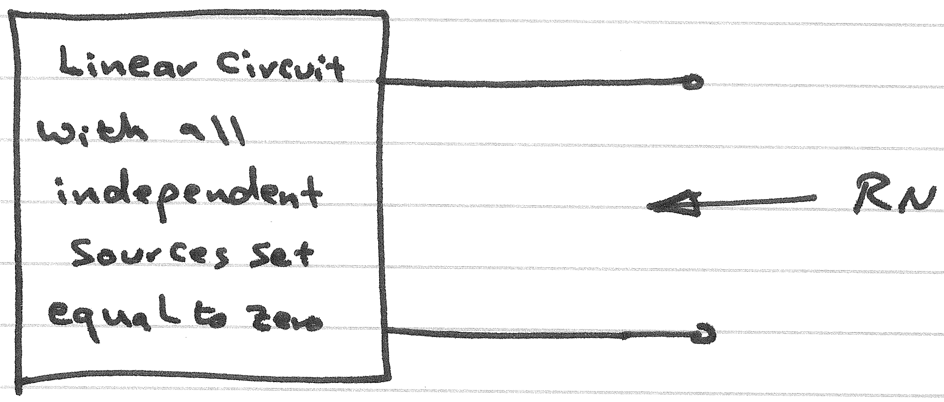
- $R_N = R_{TH}$

Norton's Theorem

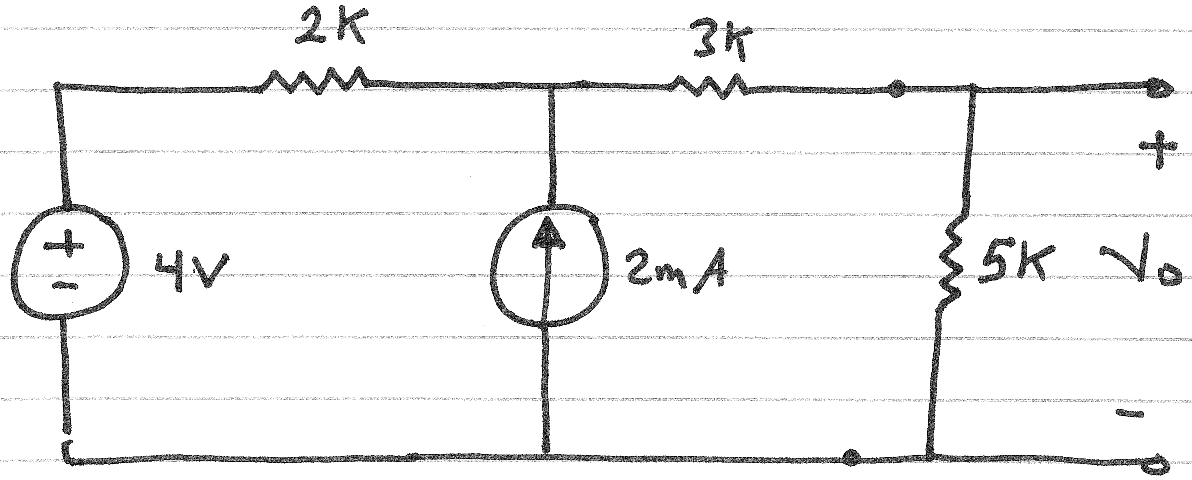
- How to find I_N

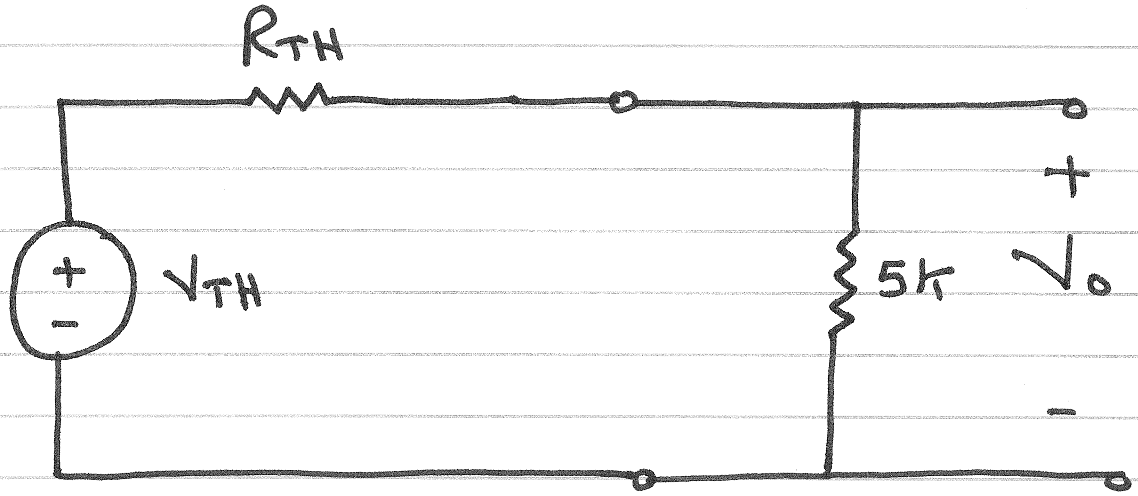


- How to find $R_N = R_{TH}$

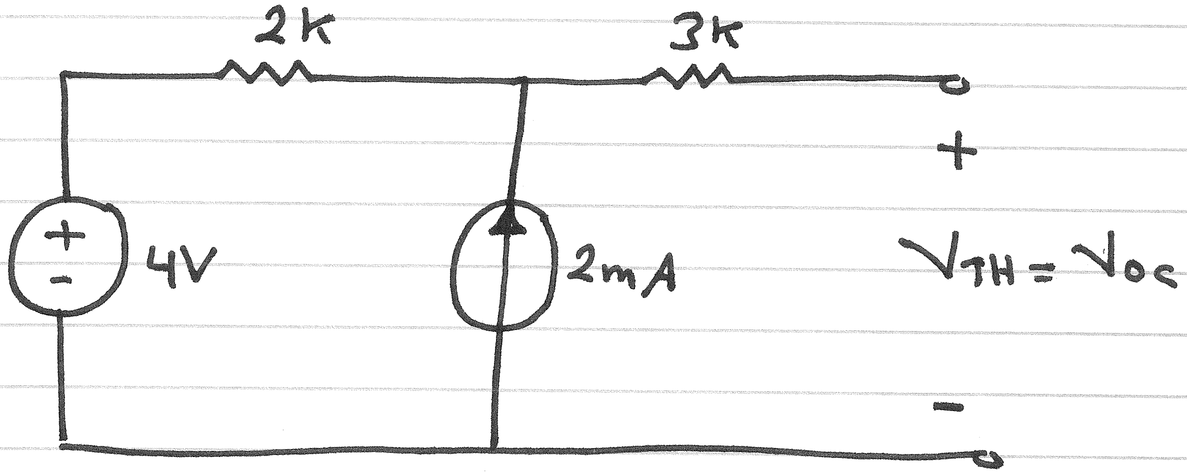


Find V_o using thevenin's theorem



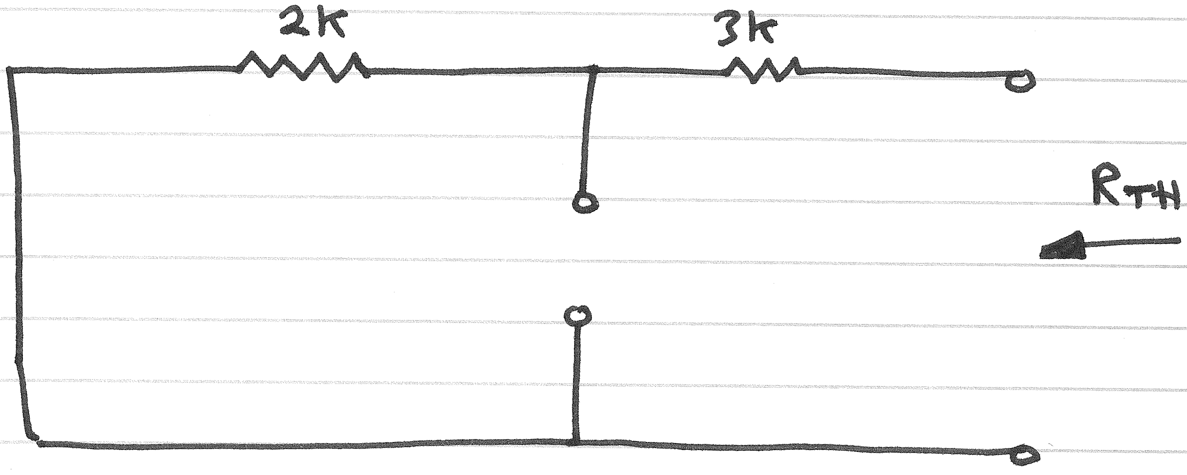


1) To find V_{TH}



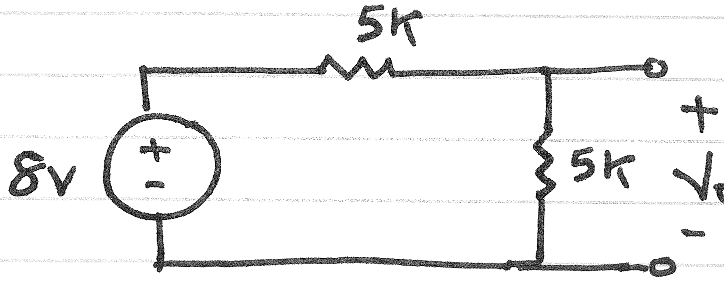
$$V_{TH} = (2K)(2mA) + 4 = 8V$$

2) To find R_{TH}



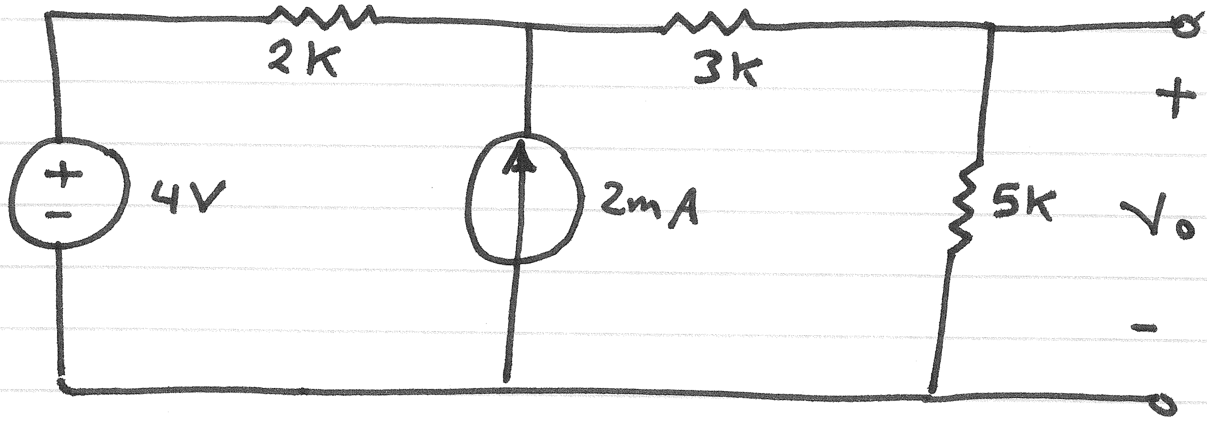
$$R_{TH} = 3K + 2K = 5K\Omega$$

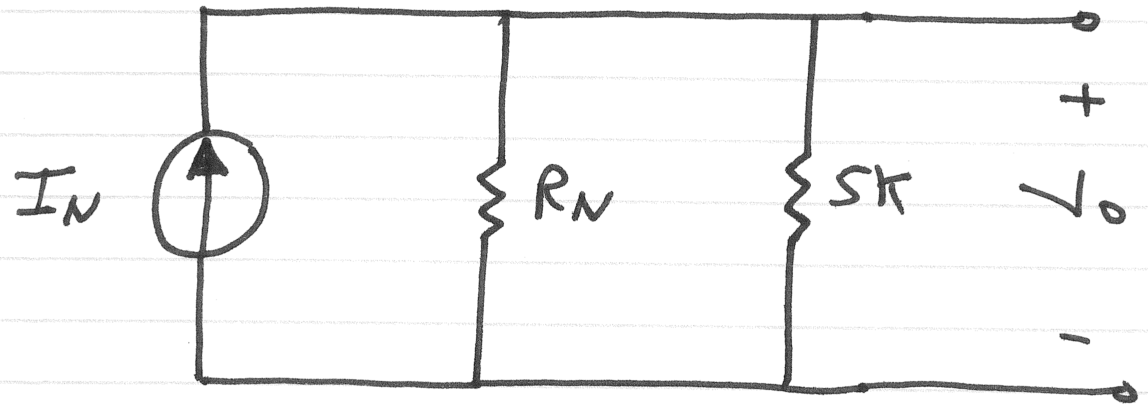
3) To find V_o



$$V_o = \frac{5}{5+5} \cdot 8 = 4V$$

Find V_o using Norton's theorem

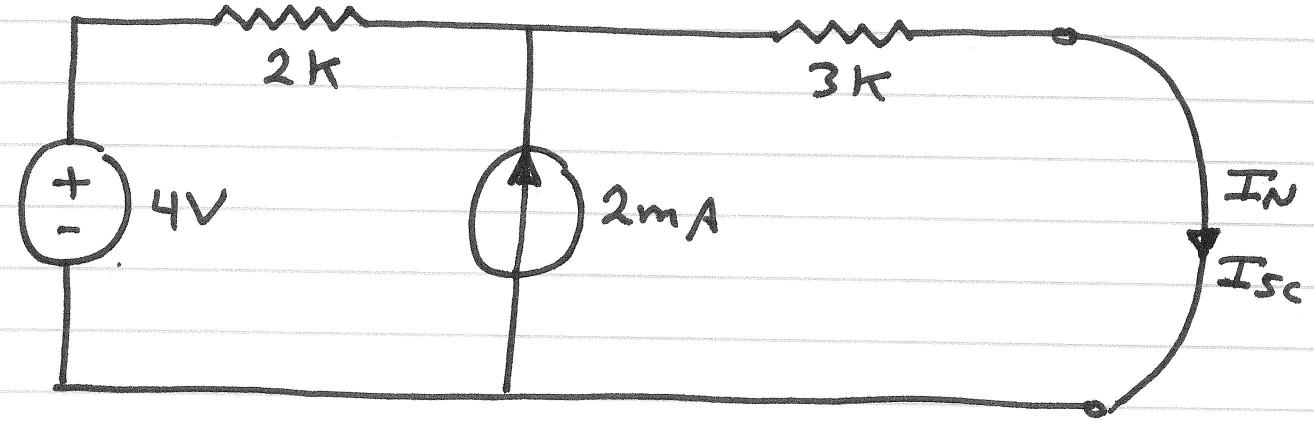


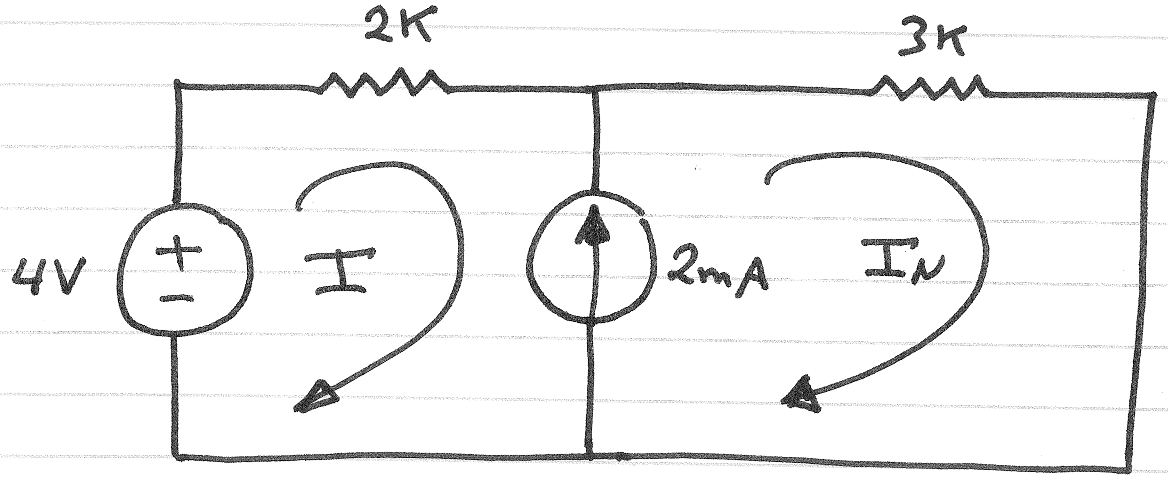


$$V_o = (R_N \| 5k)\, I_N$$

1) To find I_N

$I_N = I_{sc}$





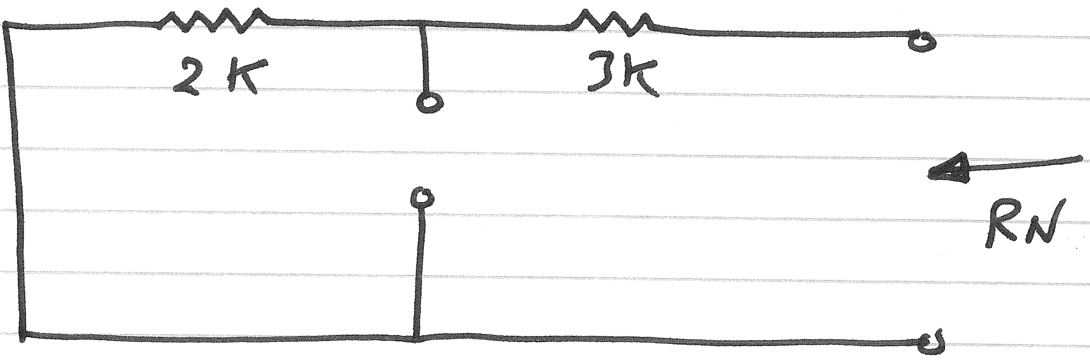
$2mA = I_N - I$ Constrain equation

$4 = (2K)I + (3K)I_N$ Supermesh equation

$\therefore \quad I_N = 1.6mA$

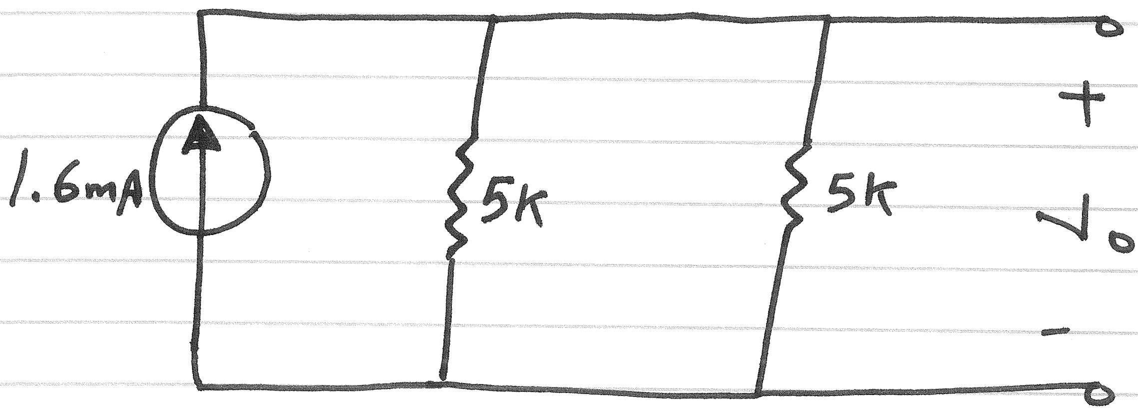
2) To find $R_N = R_{TH}$

turn off all the independent sources



$R_N = 3K + 2K = 5K$

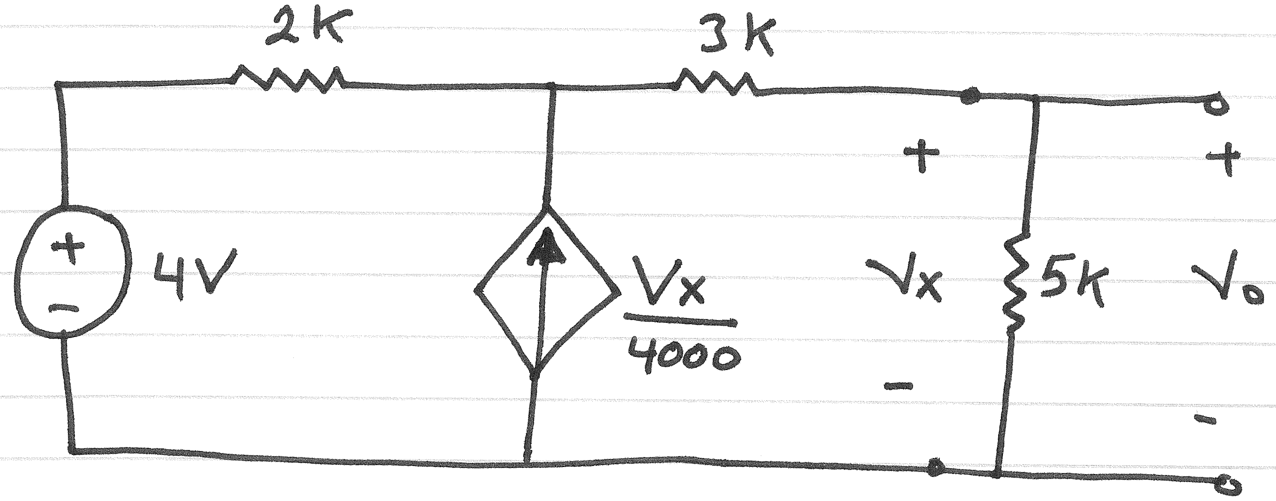
3) To find V_o

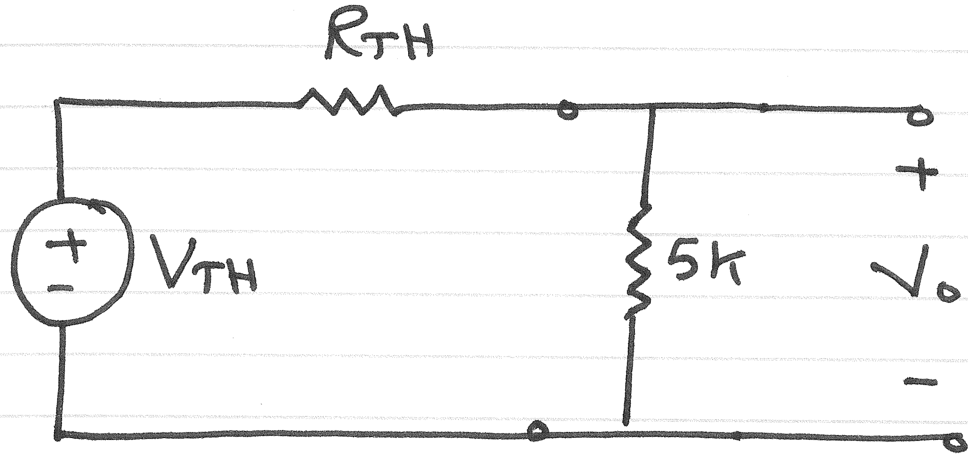


$V_o = (5K \| 5K)(1.6mA)$

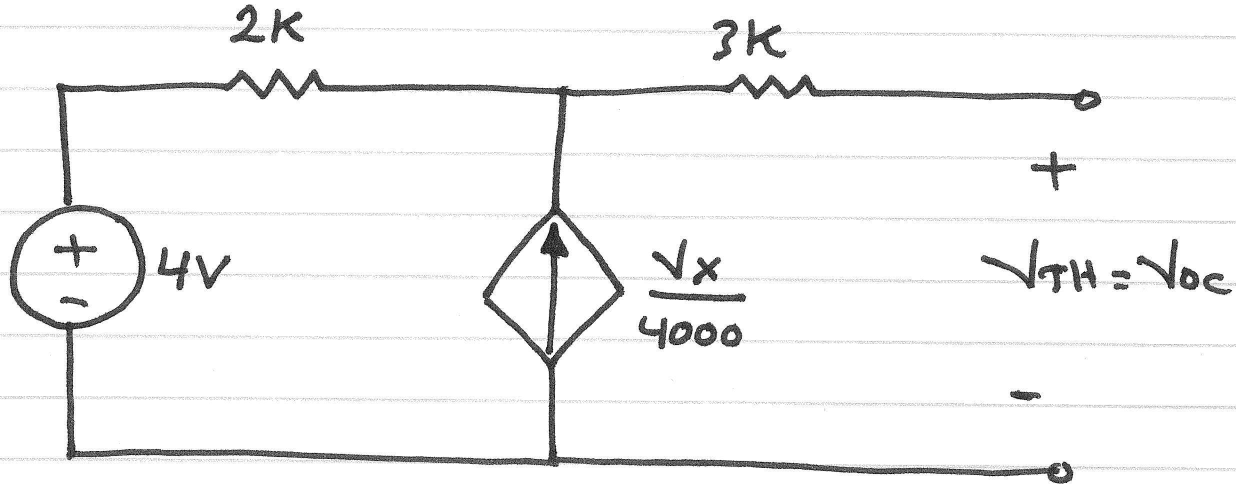
$V_o = 4V$

Find V_o using thevenin's theorem





1) To find V_{TH}



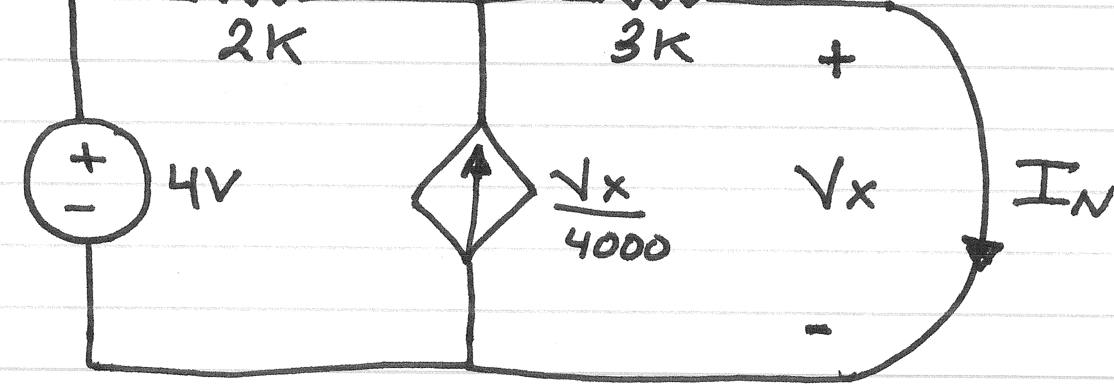
$$V_{TH} = (2K)\left(\frac{V_x}{4000}\right) + 4$$

$$V_x = V_{TH}$$

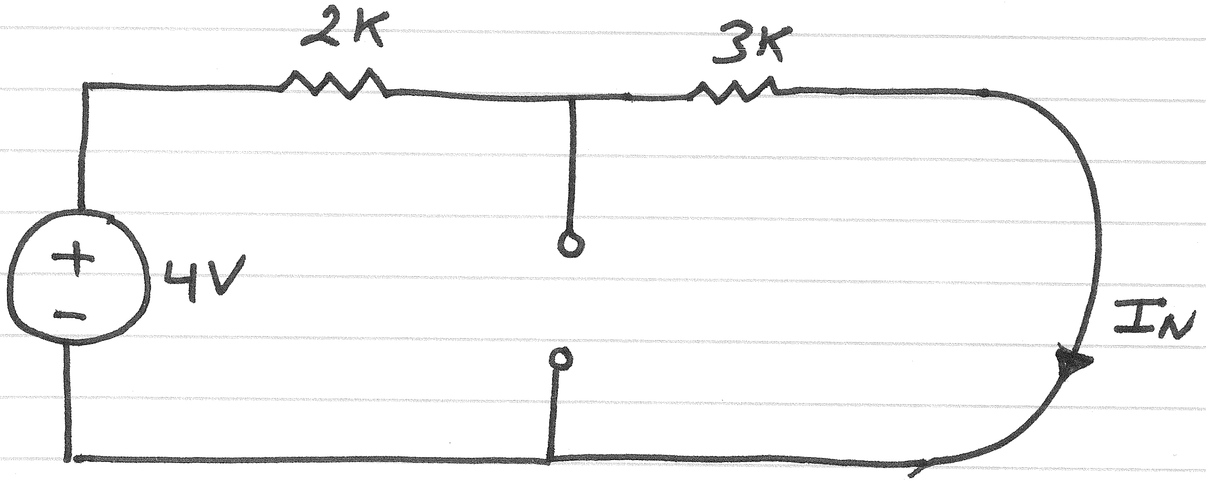
$$\therefore \quad V_{TH} = 8V$$

2) To find R_{TH}

a) method 1 : $R_{TH} = \dfrac{V_{TH}}{I_N}$



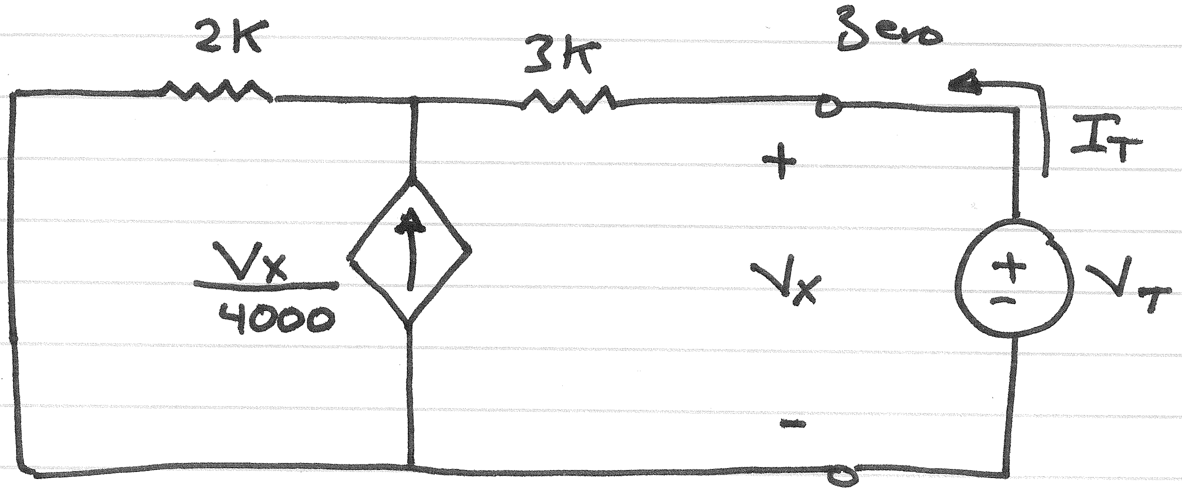
$$V_x = 0 \longrightarrow \frac{V_x}{4000} = 0 \longrightarrow \text{open Circuit}$$

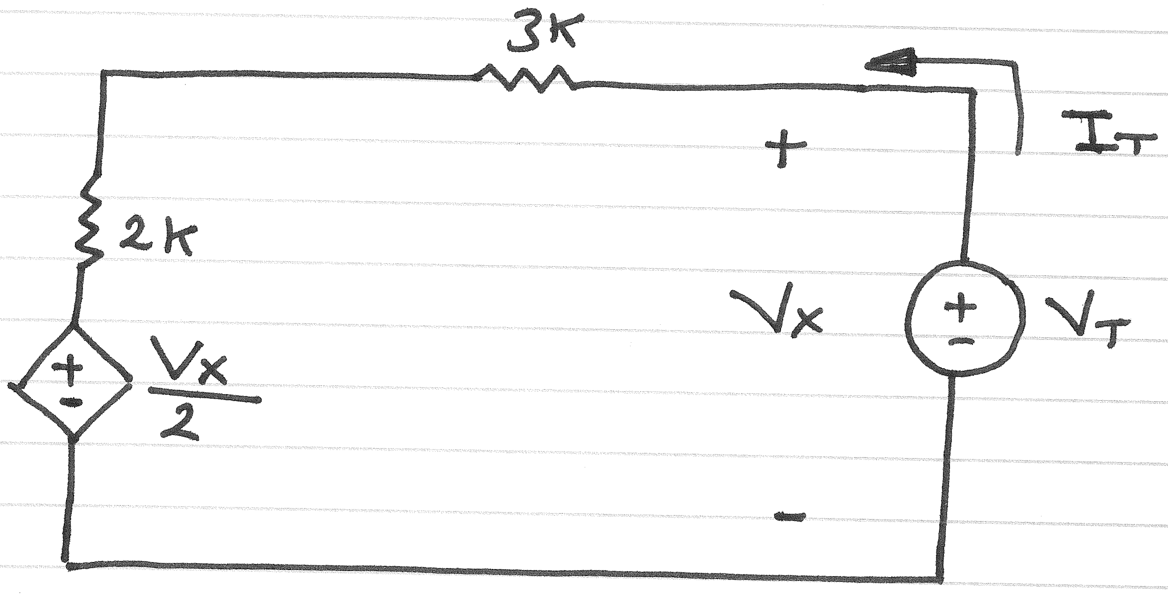


$$I_N = \frac{4V}{5K} = 0.8\,mA$$

$$\therefore\ R_{TH} = \frac{8v}{0.8mA} = 10K$$

b) method 2 : $R_{TH} = \left.\dfrac{V_T}{I_T}\right|$ all independent sources set to zero



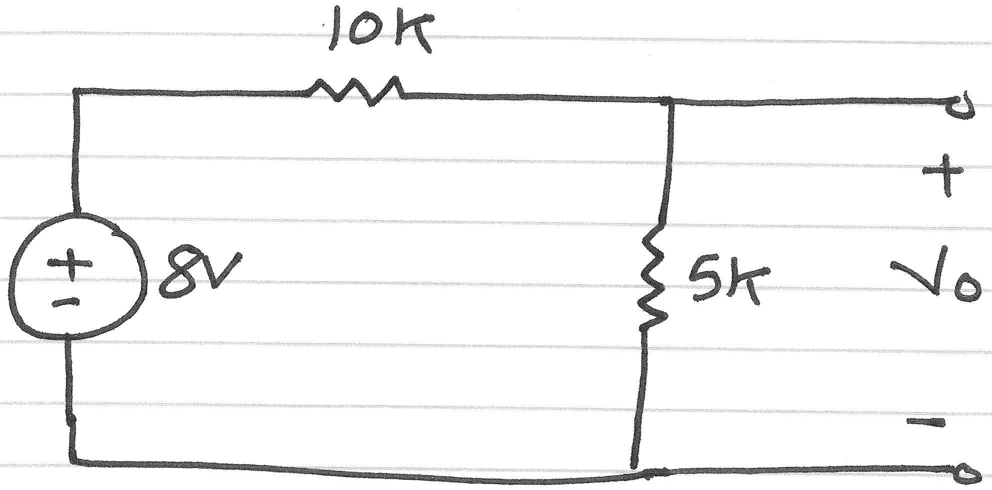


KVL :

$$-V_T + 3K\,I_T + 2K\,I_T + \frac{V_x}{2} = 0$$

$$V_x = V_T$$

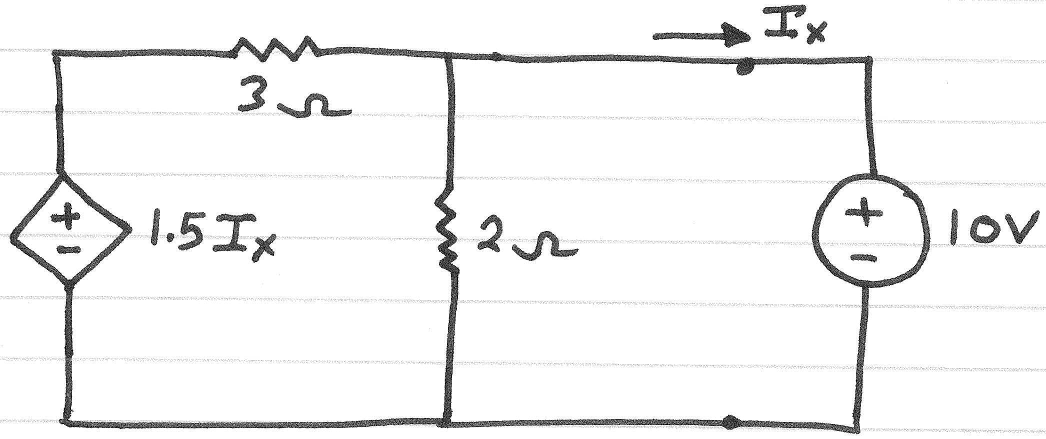
$$\therefore \quad R_{TH} = \frac{V_T}{I_T} = 10K$$

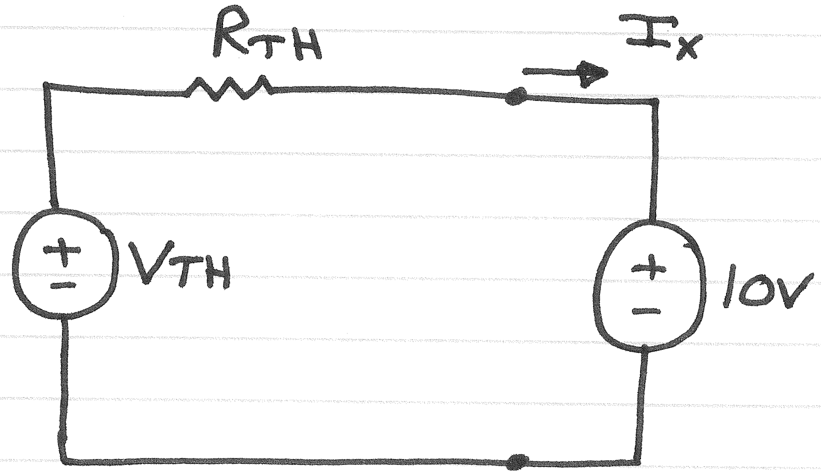


$$V_o = \frac{5K}{5K+10K}(8V)$$

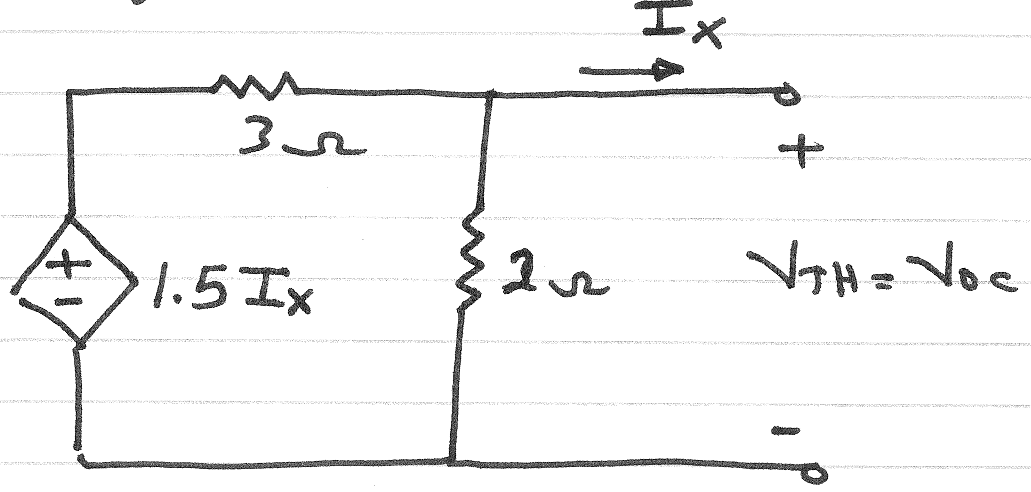
$$V_o = \frac{8}{3}\ V$$

Find I_x using thevenin's theorem





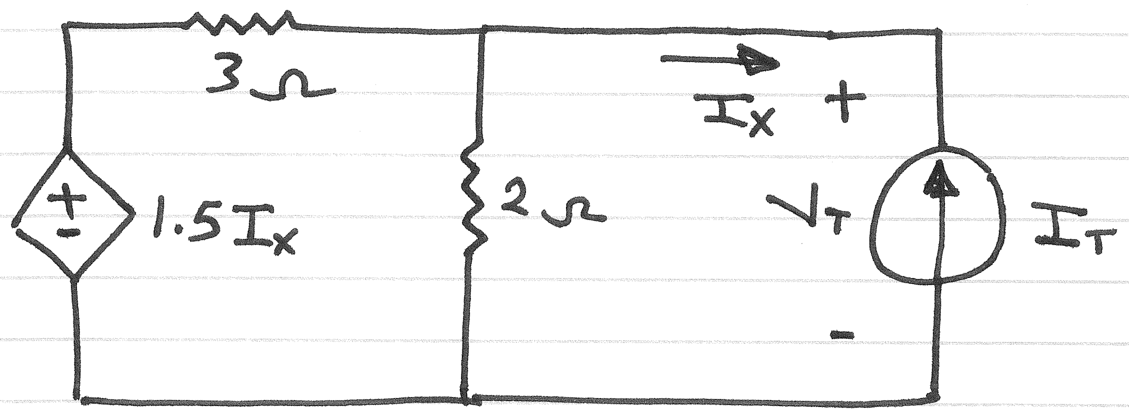
1) To find V_{TH}

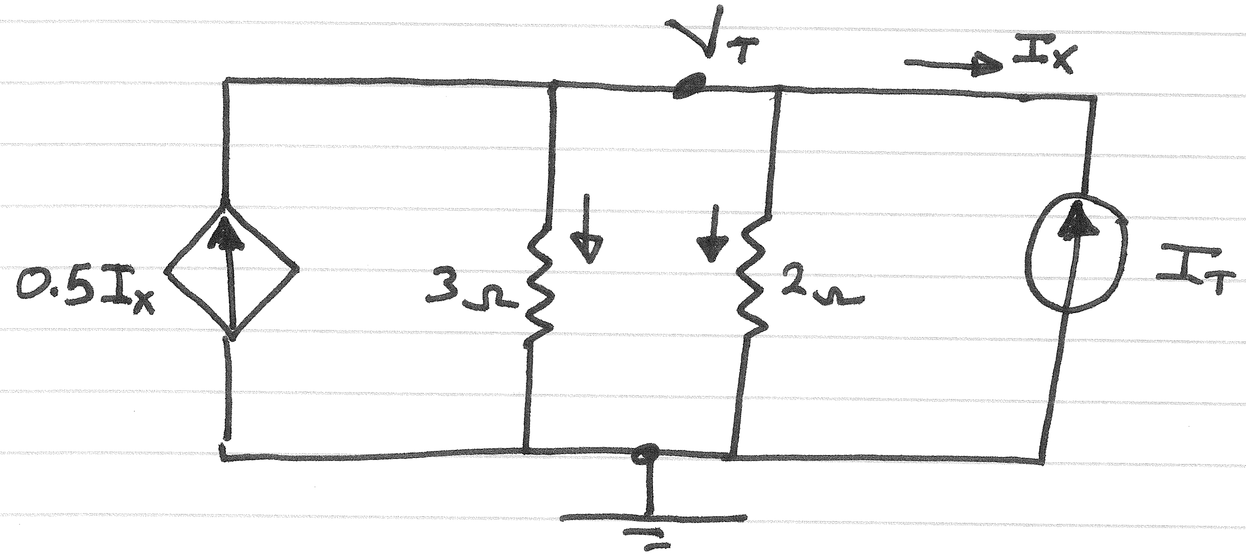


Since there is no independent sources

$\therefore\ V_{TH} = 0$

2) To find R_{TH} : $\frac{V_T}{I_T}$



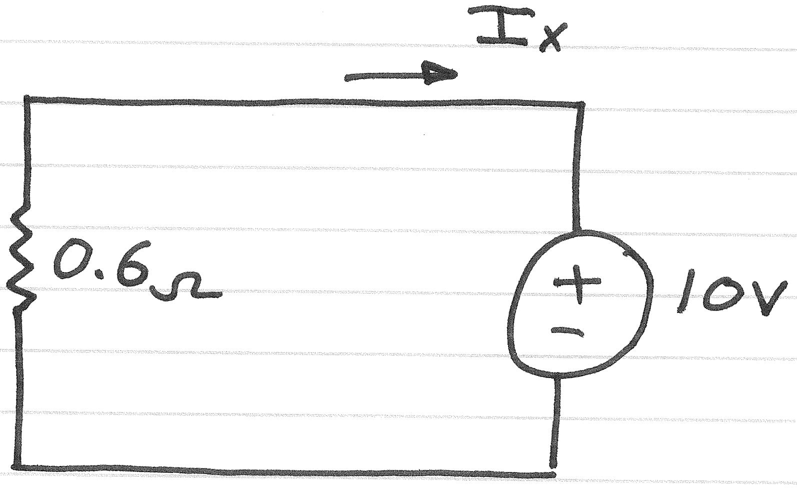


KCL :

$$0.5\,I_x + I_T = \frac{V_T}{3} + \frac{V_T}{2}$$

$$I_x = -\,I_T$$

$$\therefore\ R_{TH} = \frac{V_T}{I_T} = 0.6\,\Omega$$



$$I_x = -\frac{10}{0.6} = -16.67\,A$$

Sinusoidal Steady-state Analysis

The Sinusoidal Source

$$v_s(t) = V_m \sin \omega t$$

$V_m \equiv$ Amplitude of the sinusoid

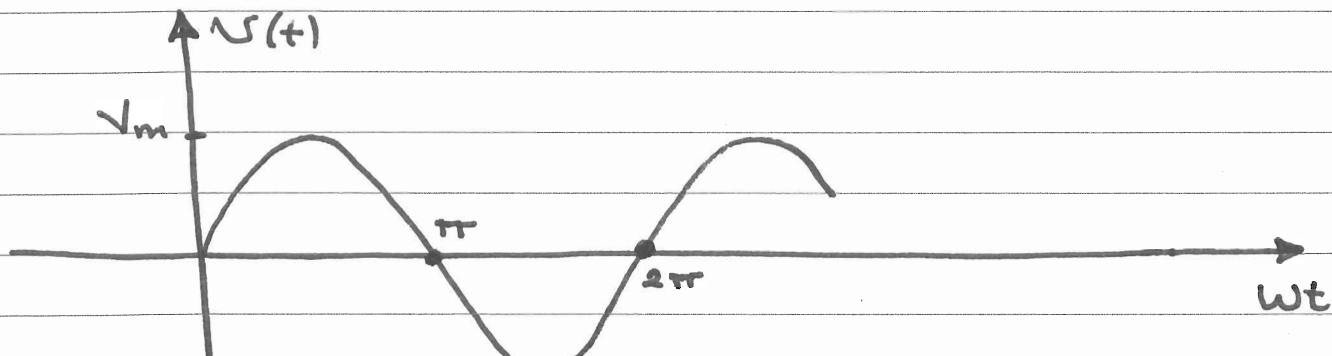
$\omega \equiv$ Angular frequency in radian/s

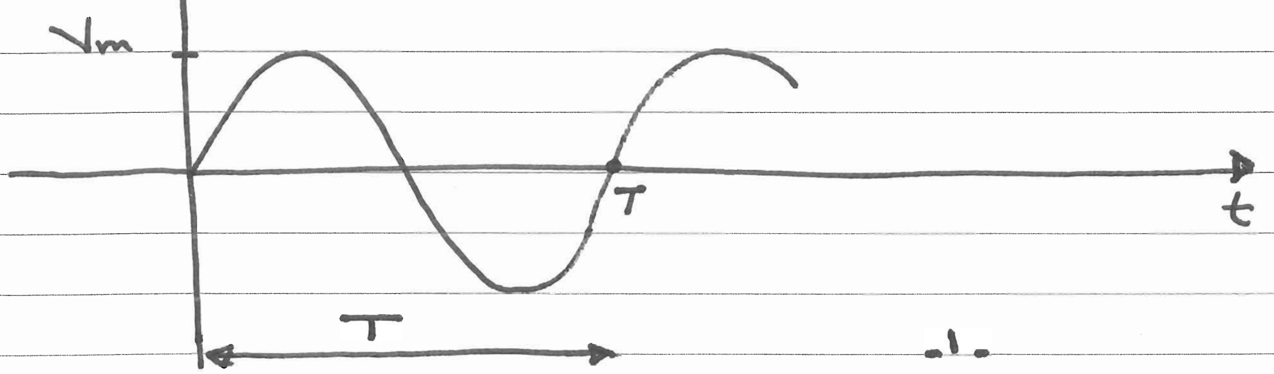
$$\omega = 2\pi f$$

$f \equiv$ frequency in Hertz

$$f = \frac{1}{T}$$

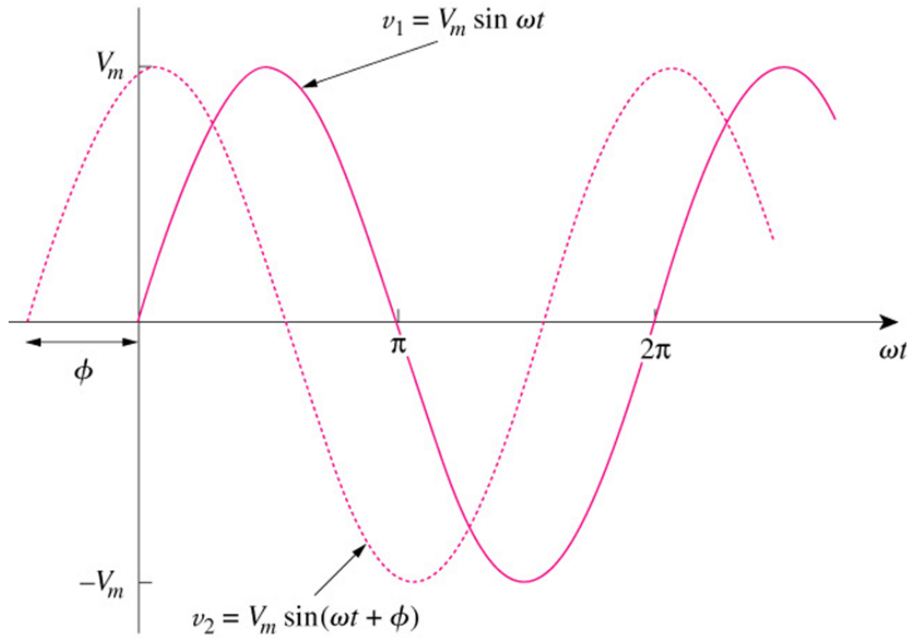
$T \equiv$ Period in seconds





Phase of Sinusoids

➢ Consider the sinusoidal voltage having phase φ, $v(t) = V_m \sin(\omega t + \phi)$



- v_2 LEADS v_1 by phase φ.
- v_1 LAGS v_2 by phase φ.
- v_1 and v_2 are out of phase.

Phase of Sinusoids

The terms Lead and Lag are used to indicate the relationship between two sinusoidal wave forms of the same frequency plotted on the same set of axes.

$v_1(t) = V_m \sin \omega t$

$v_2(t) = V_m \sin(\omega t + \theta)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by θ

or

$v_1(t)$ Lags $v_2(t)$ by θ

Trigonometric Identities

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

$$\cos(A \pm B) = \cos A \cos B \mp \sin A \sin B$$

$$\sin(\omega t \pm 180^\circ) = -\sin \omega t$$

$$\cos(\omega t \pm 180^\circ) = -\cos \omega t$$

$$\sin(\omega t \pm 90^\circ) = \pm \cos \omega t$$

$$\cos(\omega t \pm 90^\circ) = \mp \sin \omega t$$

$$A \cos \omega t + B \sin \omega t = C \cos(\omega t - \theta)$$

where

$$C = \sqrt{A^2 + B^2} \quad \text{and} \quad \theta = \tan^{-1} \frac{B}{A}$$

Let $v_1(t) = 10 \sin(5t - 30^\circ)$

$v_2(t) = 15 \sin(5t + 10^\circ)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by 40°

Let $i_1(t) = 2 \sin(377t + 45^\circ)$

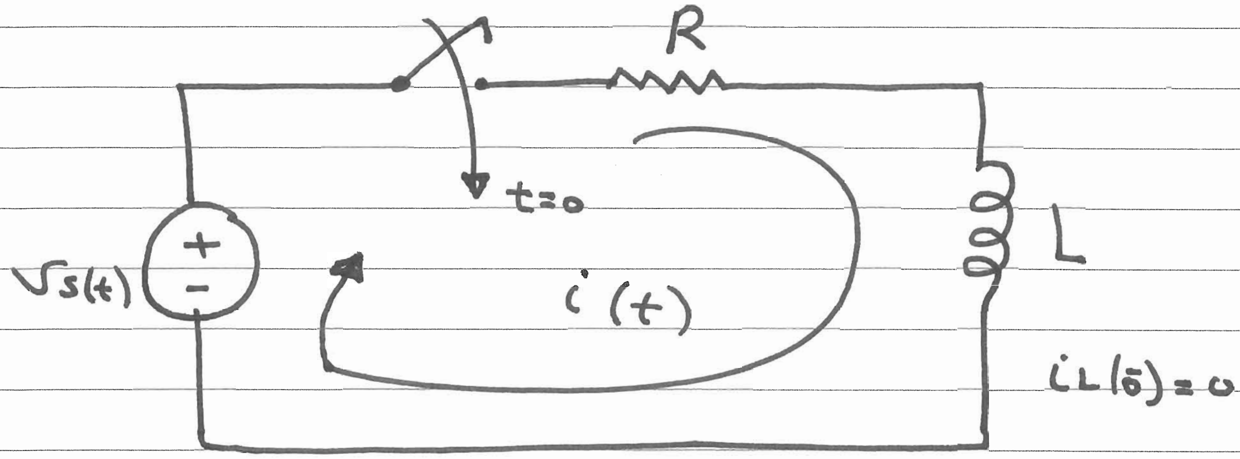
$i_2(t) = 0.5 \cos(377t + 10^\circ)$

$\cos\alpha = \sin(\alpha + 90^\circ)$

$0.5 \cos(377t + 10^\circ) = 0.5 \sin(377t + 100^\circ)$

$\therefore$ $i_2(t)$ leads $i_1(t)$ by 55°

The Sinusoidal Response



Find $i(t)$ for $t > 0$

given $v_s(t) = V_m \cos \omega t$ V

KVL :

$$v_s(t) = R i(t) + L \frac{di(t)}{dt}$$

$$V_m \cos \omega t = R i(t) + L \frac{di(t)}{dt}$$

First order nonhomogenouse differential equation

$$\therefore \; i(t) = i_n(t) + i_f(t)$$

$$i_n(t) = A e^{-t/\tau} + i_f(t)$$

$$i_f(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

To find I_1 and I_2

$$V_m \cos \omega t = R i(t) + L \frac{di(t)}{dt}$$

$$V_m \cos \omega t = R \left[I_1 \cos \omega t + I_2 \sin \omega t \right]$$

$$+ L\omega \left[- I_1 \sin \omega t + I_2 \cos \omega t \right]$$

Collect the Cosine and sine terms

$$0 = (- L I_1 \omega + R I_2) \sin \omega t + (L I_2 \omega + R I_1 - V_m) \cos \omega t$$

$\therefore$

$$- \omega L I_1 + R I_2 = 0$$

$$\omega L I_2 + R I_1 - V_m = 0$$

$$I_1 = \frac{R V_m}{R^2 + \omega^2 L^2}$$

$$I_2 = \frac{\omega L V_m}{R^2 + \omega^2 L^2}$$

$$\therefore i_f(t) = \frac{R V_m}{R^2 + \omega^2 L^2} \cos \omega t + \frac{\omega L V_m}{R^2 + \omega^2 L^2} \sin \omega t$$

$$i_f(t) = I_1 \cos \omega t + I_2 \sin \omega t$$

$$i_f(t) = C \cos(\omega t - \phi)$$

$$i_f(t) = C \cos \omega t \cos \phi + C \sin \omega t \sin \phi$$

$$\therefore \quad I_1 = C \cos \phi$$

$$I_2 = C \sin \phi$$

$$\frac{I_2}{I_1} = \tan \phi$$

$$\therefore \phi = \tan^{-1} \frac{I_2}{I_1} = \tan^{-1} \frac{\omega L}{R} \qquad \text{—— (1)}$$

$$I_1^2 + I_2^2 = C^2 \cos^2 \phi + C^2 \sin^2 \phi$$

$$I_1^2 + I_2^2 = C^2$$

$$\therefore C = \sqrt{I_1^2 + I_2^2}$$

$$C = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \qquad \text{—— (2)}$$

$$\therefore \quad i_f(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos\left(\omega t - \tan^{-1} \frac{\omega L}{R}\right)$$

$$\therefore\ i(t) = A e^{-t/\tau} + \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(\omega t - \tan^{-1}\frac{\omega L}{R}\right)$$

$$i(0^+) = A + \frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right) = 0$$

$$\therefore\ A = -\frac{V_m}{\sqrt{R^2+\omega^2L^2}} \cos\left(-\tan^{-1}\frac{\omega L}{R}\right)$$

$$\therefore\ i(t) = i_n(t) + i_f(t)$$

$i(t)$ = transient Componet + Steady-state Component

* The steady-state solution is a sinusoidal function with the same frequency as the source signal.

Complex Numbers

A complex number may be written in three forms

1) Rectangular Form

$$Z = x + jy$$

$j = \sqrt{-1}$, $x = Re(Z)$, $y = Im(Z)$

2) Exponential Form

$$Z = |Z| e^{j\theta}$$

$|Z|$ = Magnitude , θ = angle

3) Polar Form

$$Z = |Z| \underline{\angle \theta}$$

Euler's Law

$$e^{j\theta} = \cos\theta + j\sin\theta$$

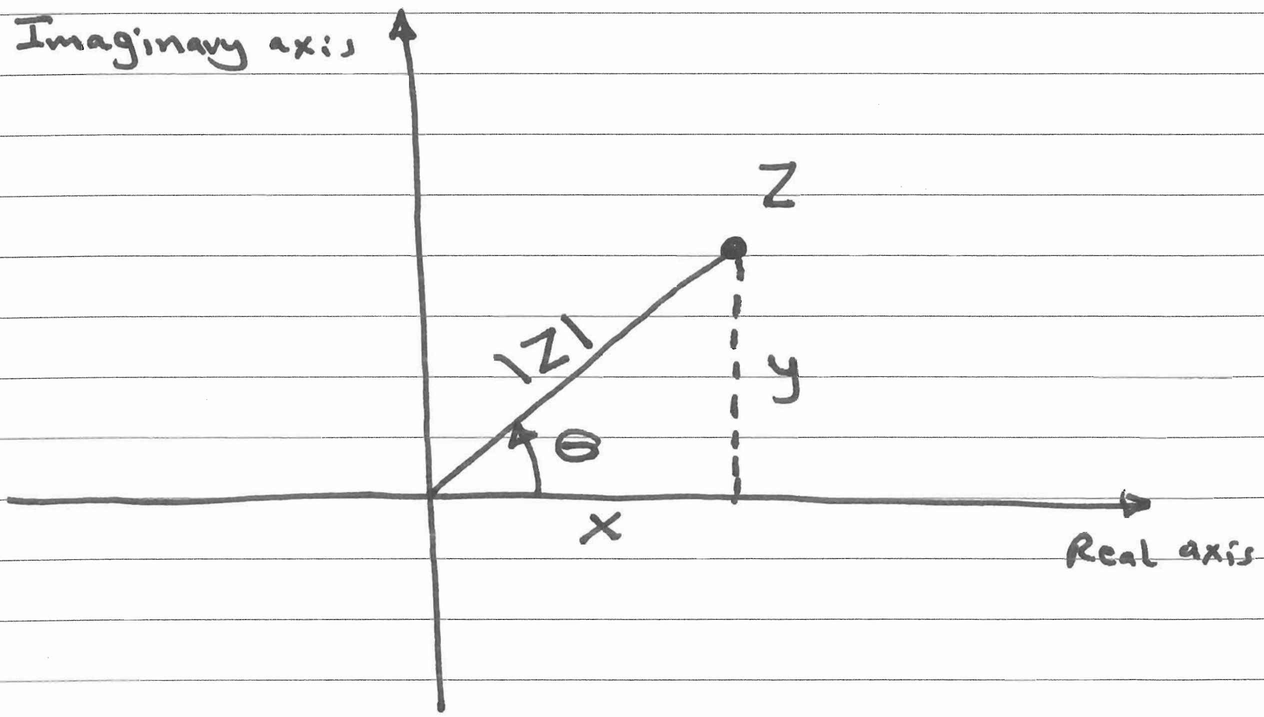
$$Z = |Z|\,e^{j\theta}$$

$$Z = |Z|\cos\theta + j|Z|\sin\theta$$

$$Z = x + jy$$

$$\therefore\ x = |Z|\cos\theta$$

$$y = |Z|\sin\theta$$



Mathematical Operations of Complex numbers

Addition : $Z_1 + Z_2 = (x_1 + x_2) + j(y_1 + y_2)$

Subtraction : $Z_1 - Z_2 = (x_1 - x_2) + j(y_1 - y_2)$

Multiplication : $Z_1 Z_2 = |Z_1||Z_2| \angle \theta_1 + \theta_2$

Division : $\frac{Z_1}{Z_2} = \frac{|Z_1|}{|Z_2|} \angle \theta_1 - \theta_2$

Complex Conjugate : $Z^* = x - jy$

$= |Z| \angle -\theta$

$$x = |Z| \cos\theta$$

$$y = |Z| \sin\theta$$

$$x^2 + y^2 = |Z|^2 \cos^2\theta + |Z| \sin^2\theta$$

$$x^2 + y^2 = |Z|^2 \left(\cos^2\theta + \sin^2\theta \right)$$

$$x^2 + y^2 = |Z|^2$$

$$\therefore |Z| = \sqrt{x^2 + y^2}$$

$$\frac{y}{x} = \frac{|Z| \sin\theta}{|Z| \cos\theta} = \tan\theta$$

$$\therefore \theta = \tan^{-1} \frac{y}{x}$$

$$Z_1 = 4 + j3 = 5\angle 36.9^\circ$$

$$Z_2 = 3 + j4 = 5\angle 53.1^\circ$$

$$Z_1 + Z_2 = 7 + j7$$

$$Z_1 - Z_2 = 1 - j1$$

$$Z_1 Z_2 = 5\angle 36.9^\circ \cdot 5\angle 53.1^\circ = 25\angle 90^\circ$$

$$\frac{Z_1}{Z_2} = \frac{5\angle 36.9^\circ}{5\angle 53.1^\circ} = 1\angle -16.2^\circ$$

or

$$Z_1 Z_2 = (4+j3)(3+j4)$$

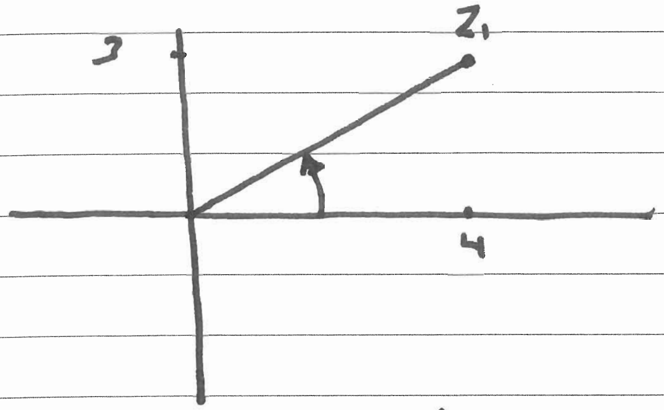
$$= 12 + j16 + j9 - 12$$

$$Z_1 Z_2 = j25$$

$$\frac{Z_1}{Z_2} = \frac{4+j3}{3+j4} \cdot \frac{3-j4}{3-j4} = \frac{12 - j16 + j9 + 12}{25}$$

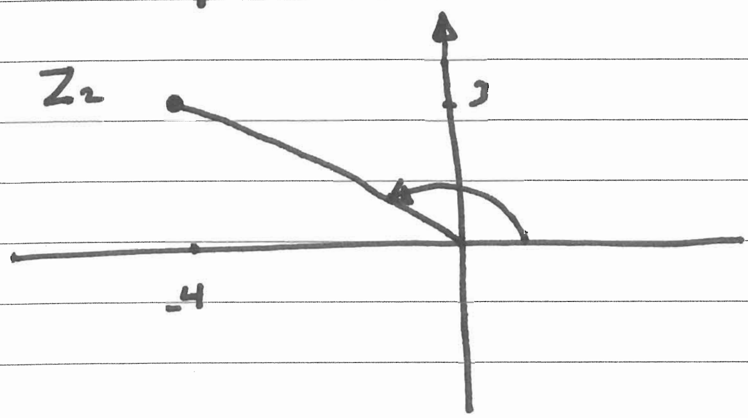
$$= \frac{24 - j5}{25} = \frac{24}{25} - j\frac{5}{25}$$

The graphical Representation



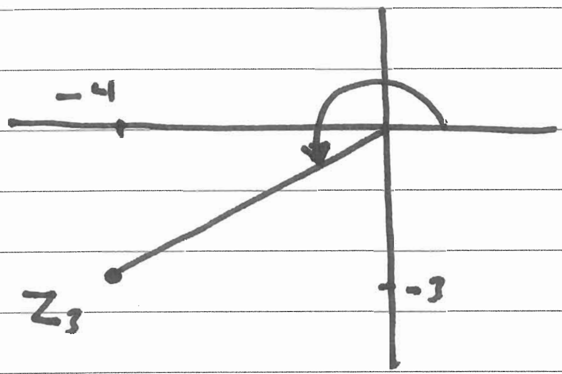
$Z_1 = 4 + j3$

$Z_1 = 5 \angle 36.9^\circ$



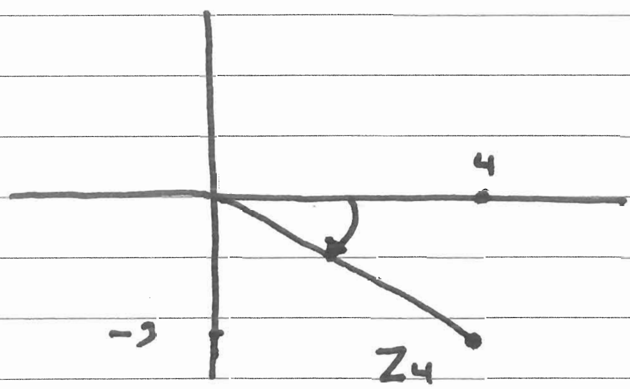
$Z_2 = -4 + j3$

$Z_2 = 5 \angle 143.1$



$Z_3 = -4 - j3$

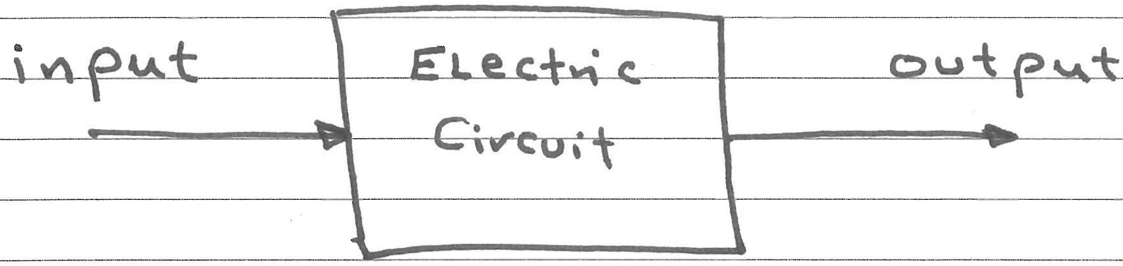
$Z_3 = 5 \angle 216.9$



$Z_4 = 4 - j3$

$Z_4 = 5 \angle -36.9^\circ$

The phasor Concept



$$V_m \cos(\omega t+\theta) \longrightarrow I_m \cos(\omega t+\phi)$$

$$V_m \sin(\omega t+\theta) \longrightarrow I_m \sin(\omega t+\phi)$$

$$j V_m \sin(\omega t+\theta) \longrightarrow j\, I_m \sin(\omega t+\phi)$$

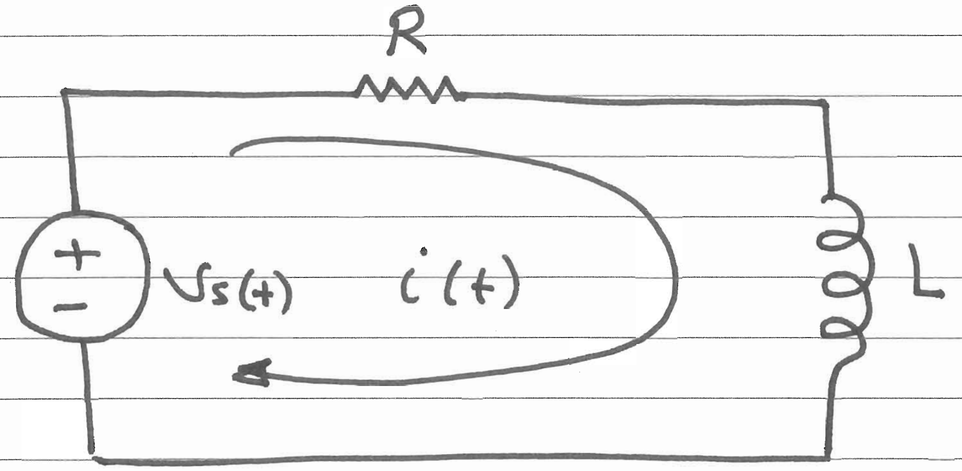
$$\begin{matrix} V_m \cos(\omega t+\theta) \\ + \\ j V_m \sin(\omega t+\theta) \end{matrix} \Longrightarrow \begin{matrix} I_m \cos(\omega t+\phi) \\ + \\ j\, I_m \sin(\omega t+\phi) \end{matrix}$$

$$V_m e^{j(\omega t+\theta)} \longrightarrow I_m e^{j(\omega t+\phi)}$$

Instead of Applying a real forcing function to obtain the desired real response, we apply a Complex forcing function whose real part is the given real forcing function.

We obtain a Complex response whose real part is the desired real response.

Sinusoidal and Complex forcing function



$$v_s(t) = V_m \cos\omega t$$

$$\therefore \quad i(t) = I_m \cos(\omega t + \phi)$$

$$v_s(t) \longrightarrow V_m e^{j\omega t}$$

$$i(t) \longrightarrow I_m e^{j(\omega t + \phi)}$$

KVL :

$$v_s(t) = R\,i(t) + L\frac{d\,i(t)}{dt}$$

$$V_m e^{j\omega t} = R I_m e^{j(\omega t+\phi)} + j\omega L I_m e^{j(\omega t+\phi)}$$

a Complex algebraic equation

To find I_m and ϕ ; devide by $e^{j\omega t}$

$$V_m = R I_m e^{j\phi} + j\omega L I_m e^{j\phi}$$

$$V_m = I_m e^{j\phi} (R + j\omega L)$$

$$\therefore I_m e^{j\phi} = \frac{V_m}{R + j\omega L}$$

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2} \; e^{j \tan^{-1} \frac{\omega L}{R}}}$$

$$I_m e^{j\phi} = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} e^{-j \tan^{-1} \frac{\omega L}{R}}$$

$$\therefore I_m = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}}$$

$$\phi = -\tan^{-1} \frac{\omega L}{R}$$

$$\therefore i(t) = I_m \cos(\omega t + \phi)$$

$$i(t) = \frac{V_m}{\sqrt{R^2 + \omega^2 L^2}} \cos(\omega t + \phi)$$

Phasors

Given the sinusoids $i(t) = I_m \cos(\omega t + \phi_i)$ and $v(t) = V_m \cos(\omega t + \theta_v)$

We can obtain the phasor forms as:

$i(t) = I_m \cos(\omega t + \phi_i)$, then $\vec{I} = I_m \angle \phi_i$

$v(t) = V_m \cos(\omega t + \theta_v)$, then $\vec{V} = V_m \angle \theta_v$

$$i(t) = 6 \cos(50t - 40^\circ) \ A$$

$$\therefore \vec{I} = 6 \angle -40^\circ \ A$$

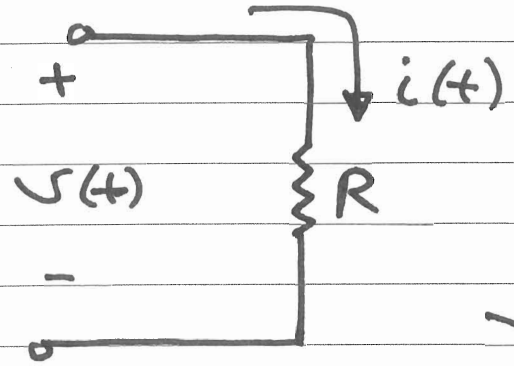
$$v(t) = -4 \sin(30t + 50^\circ) \ V$$

$$v(t) = 4 \cos(30t + 140^\circ) \ V$$

$$\therefore \vec{V} = 4 \angle 140^\circ \ V$$

Phasor Relationships for Circuit Elements

Resistor :



$$v(t) = R\,i(t)$$

$$V_m e^{j(\omega t + \theta_v)} = R\, I_m e^{j(\omega t + \phi)}$$

$$V_m e^{j\theta_v} = R\, I_m e^{j\phi_i}$$

$$V_m \angle \theta_v = R\, I_m \angle \phi_i$$

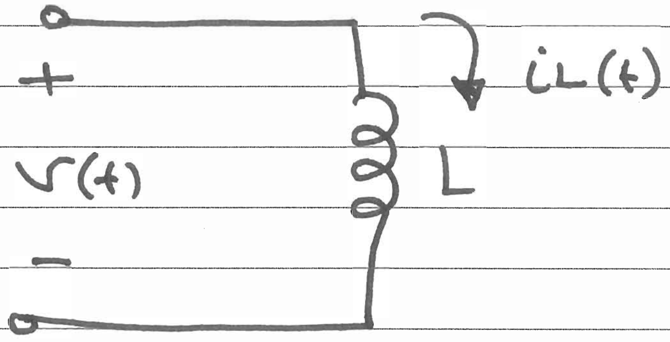
$$\boxed{\vec{V} = R\,\vec{I}}$$

$$V_m = R I_m$$

$$\theta_v = \phi_i$$

* Voltage and Current of a resistor are in phase.

Inductor :



$$v(t) = L\frac{di(t)}{dt}$$

$$V_m e^{j(\omega t+\theta_v)} = L\frac{d}{dt}\left(I_m e^{j(\omega t+\phi_i)}\right)$$

$$V_m e^{j(\omega t+\theta_v)} = j\omega L\, I_m e^{j(\omega t+\phi_i)}$$

$$V_m e^{j\theta_v} = j\omega L\, I_m e^{j\phi_i}$$

$$V_m \angle \theta_v = j\omega L\, I_m \angle \phi_i$$

$$\boxed{\vec{V} = j\omega L\, \vec{I}}$$

* $V_m = \omega L I$

$$V_m \angle \theta_v = \omega L \angle 90^\circ \cdot I_m \angle \phi_i$$

$$V_m \angle \theta_v = \omega L I_m \angle \phi_i + 90^\circ$$

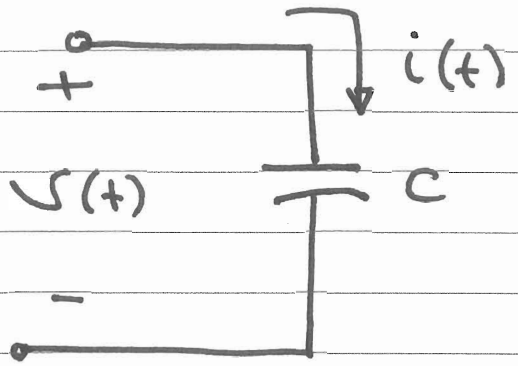
$$V_m \angle \theta_v = \omega L I_m \angle \phi_i + 90^\circ$$

$$\therefore \quad V_m = \omega L I_m$$

$$\theta_v = \phi_i + 90^\circ$$

The voltage Leads the Current by 90°

Capacitor :



$$i(t) = C \frac{dv(t)}{dt}$$

$$I_m e^{j(\omega t+\phi_i)} = C \frac{d}{dt}\left(V_m e^{j(\omega t+\theta_v)}\right)$$

$$I_m e^{j(\omega t+\phi_i)} = j\omega C V_m e^{j(\omega t+\theta_v)}$$

$$I_m e^{j\phi_i} = j\omega C V_m e^{j\theta_v}$$

$$I_m \angle \phi_i = j\omega C V_m \angle \theta_v$$

$$\boxed{\vec{I} = j\omega C \vec{V}}$$

$$I_m \angle \phi_i = \omega C \angle 90^\circ \; V_m \angle \theta_v$$

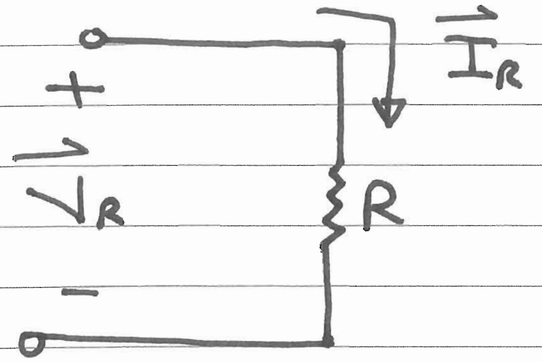
$$I_m \angle \phi_i = \omega C V_m \angle \theta_v + 90^\circ$$

$$\therefore \quad I_m = \omega C V_m$$

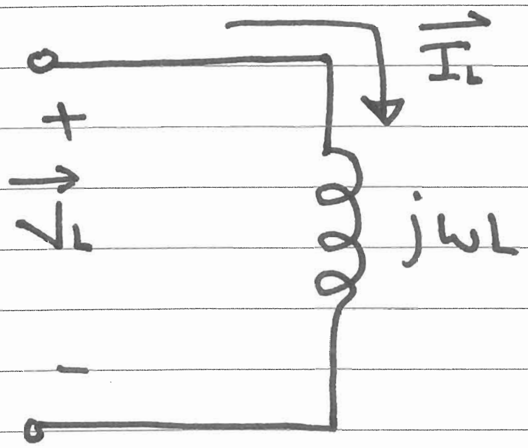
$$\phi_i = \theta_v + 90^\circ$$

The Current Leads the Voltage by 90°.

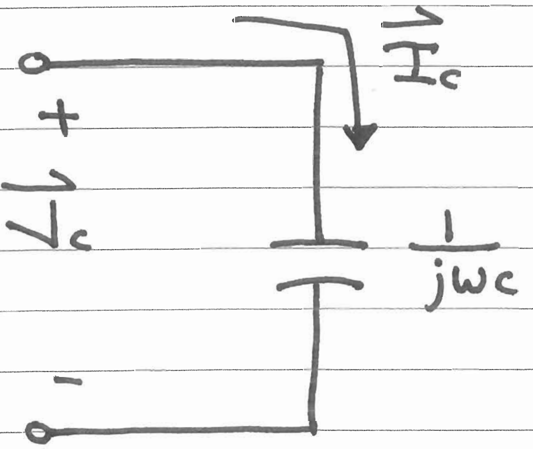
Phasor Relationships For Circuit Elements



$$\vec{V}_R = R\,\vec{I}_R$$



$$\vec{V}_L = j\omega L\,\vec{I}_L$$



$$\vec{V}_C = \frac{1}{j\omega C}\,\vec{I}_C$$

$$\vec{V} = Z(j\omega)\,\vec{I}$$

Impedance and Admittance

$$Z(j\omega) = \frac{\vec{V}}{\vec{I}} \qquad \text{Impedance}, \Omega$$

or $\vec{V} = Z(j\omega)\,\vec{I}$

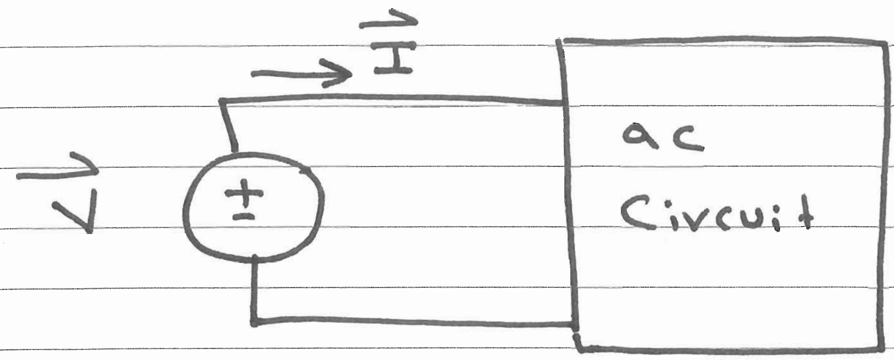
$$Y(j\omega) = \frac{\vec{I}}{\vec{V}} \qquad \text{Admittance}, \mho$$

or $\vec{I} = Y(j\omega)\,\vec{V}$

$$\therefore \quad Z(j\omega) = \frac{1}{Y(j\omega)}$$

Element	Impedance	Admittance
R	$Z(j\omega) = R$	$Y(j\omega) = \frac{1}{R}$
C	$Z(j\omega) = \frac{1}{j\omega C}$	$Y(j\omega) = j\omega C$
L	$Z(j\omega) = j\omega L$	$Y(j\omega) = \frac{1}{j\omega L}$

Impedance: $Z(j\omega)$



$$Z(j\omega) = \frac{\vec{V}}{\vec{I}} = \frac{V_m \angle \theta_v}{I_m \angle \phi_i}$$

$$Z(j\omega) = \frac{V_m}{I_m} \angle \theta_v - \phi_i$$

$$Z(j\omega) = |Z| \angle \theta_z$$

The unit of impedance is Ohm

Impedance is not a phasor but a Complex number that can be written in polar or Cartesian forms

$$\vec{Z} = R + jX$$

$R \equiv$ Resistive part

$X \equiv$ Reactive part

$$Z = |Z| \angle \theta_z$$

$$Z = R + jX$$

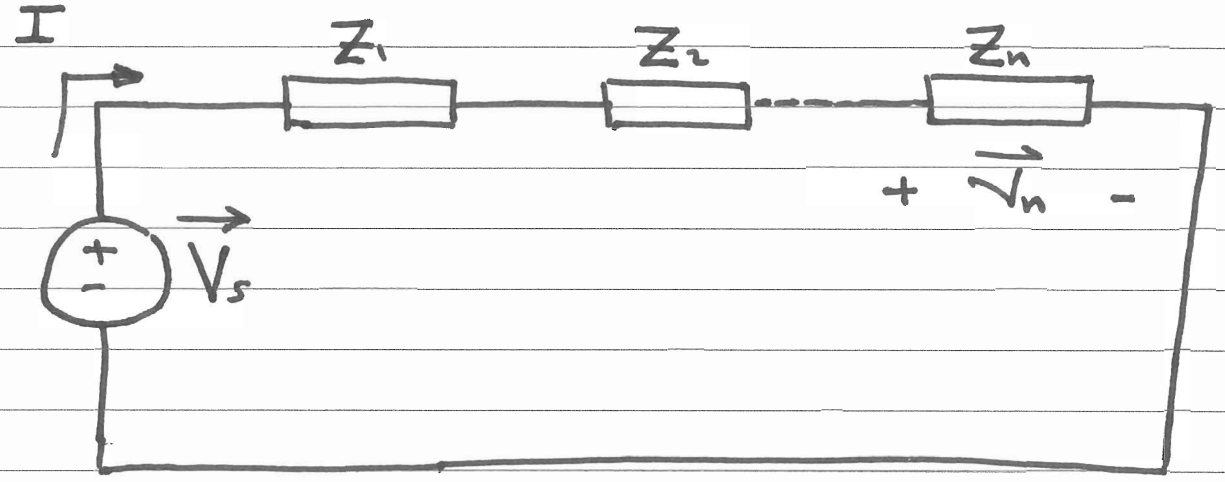
$$|Z| = \sqrt{R^2 + X^2}$$

$$\theta_z = \tan^{-1} \frac{X}{R}$$

$$X = |Z| \sin \theta_z$$

$$R = |Z| \cos \theta_z$$

Application of KVL for phasors



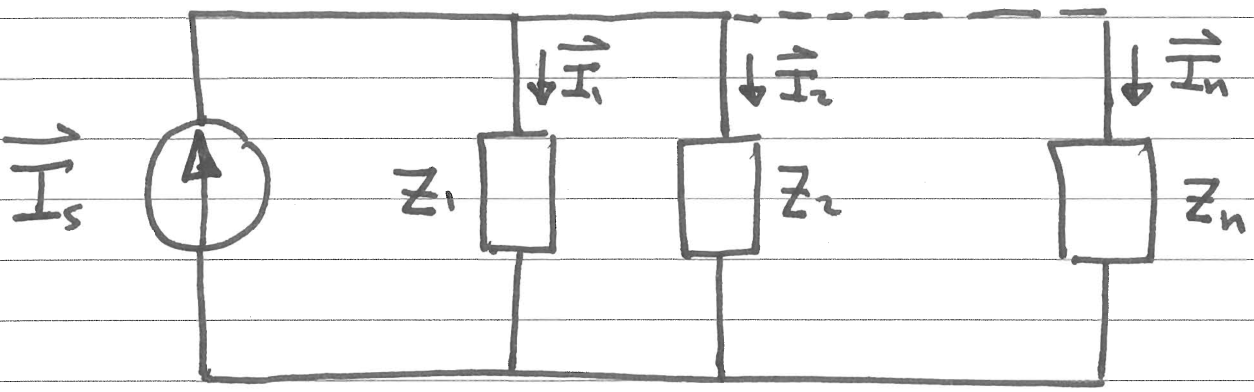
KVL : $v_s(t) = v_1(t) + v_2(t) + \cdots\cdots v_n(t)$

$$\vec{V}_s = \vec{V}_1 + \vec{V}_2 + \cdots\cdots \vec{V}_n$$

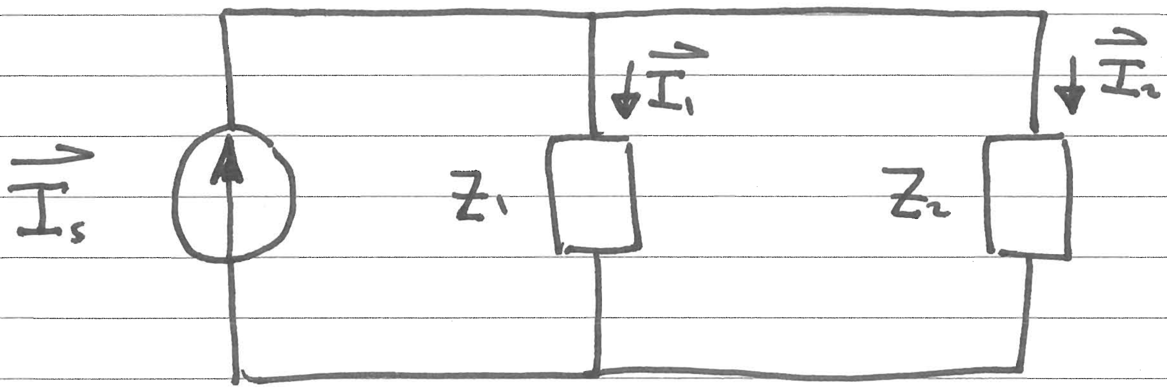
$$Z_{eq} = Z_1 + Z_2 + Z_3 + \cdots\cdots Z_n$$

$$\vec{V}_n = \frac{Z_n}{Z_1 + Z_2 + \cdots\cdots Z_n} \cdot \vec{V}_s$$

Application of KCL for phasors



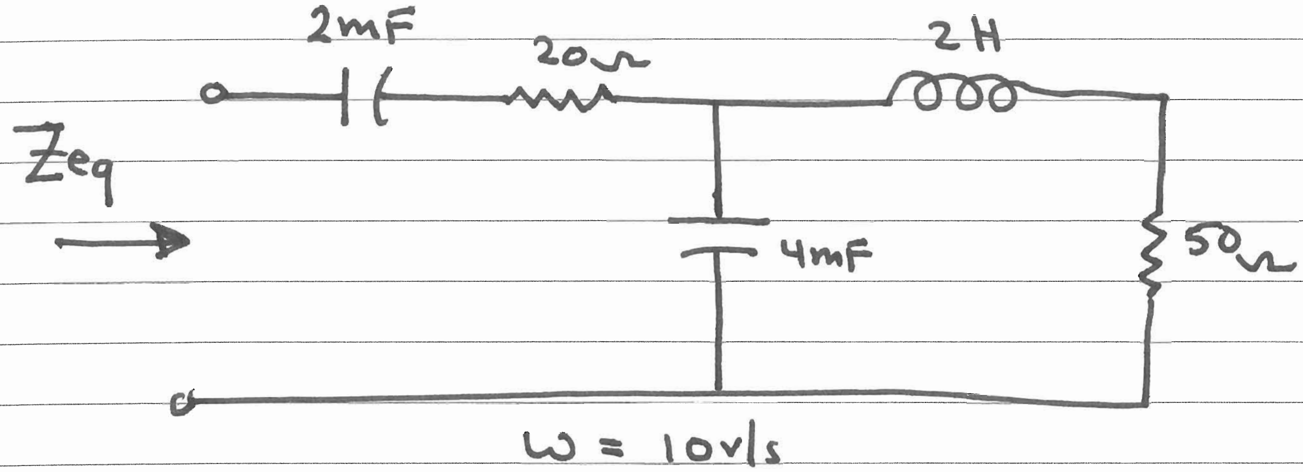
KCL : $\vec{I}_s = \vec{I}_1 + \vec{I}_2 + \cdots\cdots \vec{I}_n$



$$\vec{I}_1 = \frac{Z_2}{Z_1 + Z_2} \vec{I}_s$$

$$\vec{I}_2 = \frac{Z_1}{Z_1 + Z_2} \vec{I}_s$$

Find Zeq



$$Z_1 = 20 + \frac{1}{j(10)(2)(10^{-3})} = 20 - j50$$

$$Z_2 = 50 + j(10)(2) = 50 + j20$$

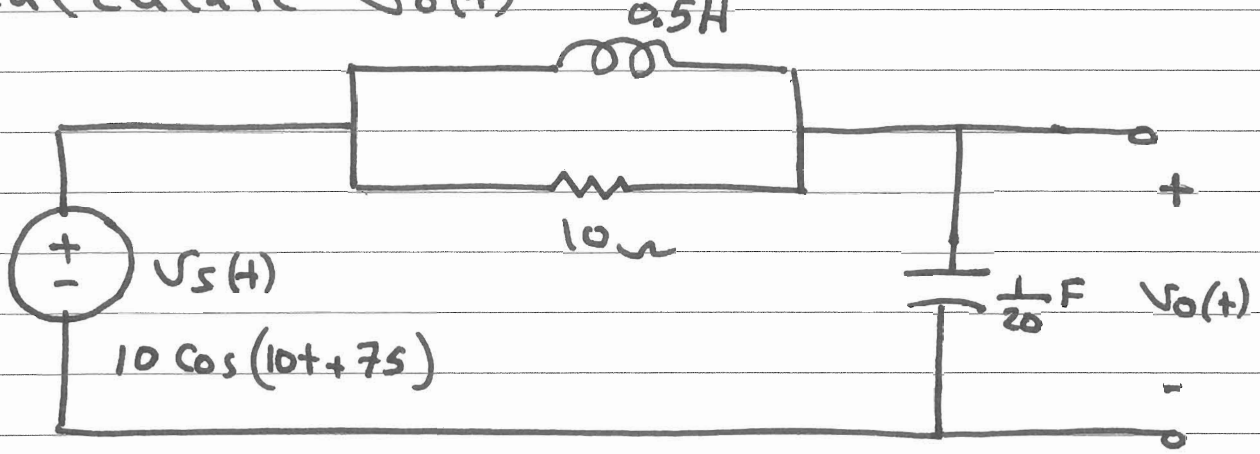
$$Z_3 = (50 + j20) \parallel \frac{1}{j(10)(4)(10^{-3})}$$

$$Z_3 = (50 + j20) \parallel -j25$$

$$Z_3 = \frac{(50 + j20)(-j25)}{50 + j20 - j25} = 12.38 - j23.76$$

$$Zeq = Z_1 + Z_3 = 32.38 - j73.76 \ \Omega$$

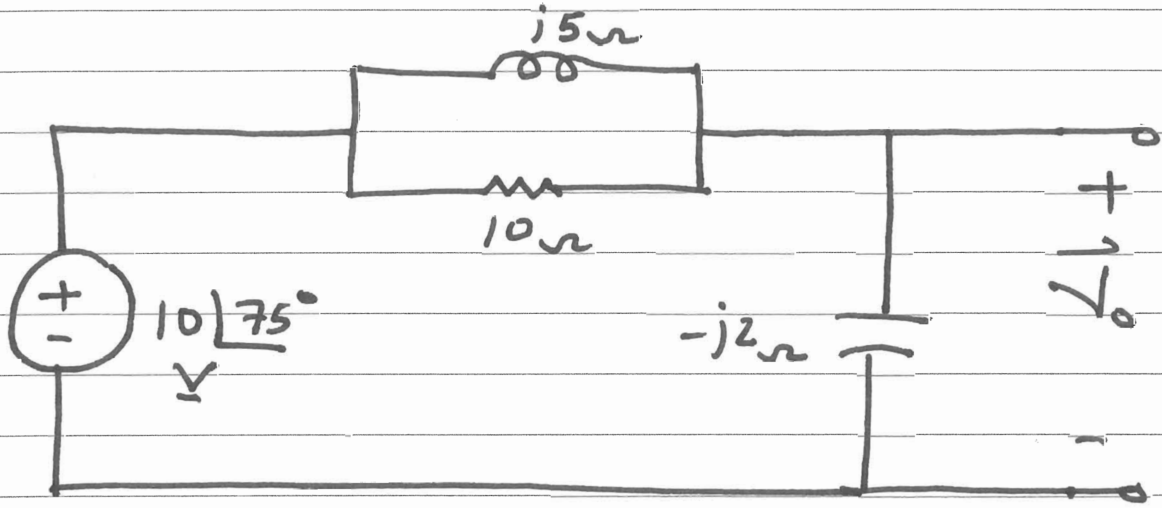
Calculate $v_o(t)$



$$Z_L(j\omega) = j\omega L = j5\ \Omega$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j2\ \Omega$$

$$\vec{V}_s = 10\angle 75^\circ\ \text{V}$$

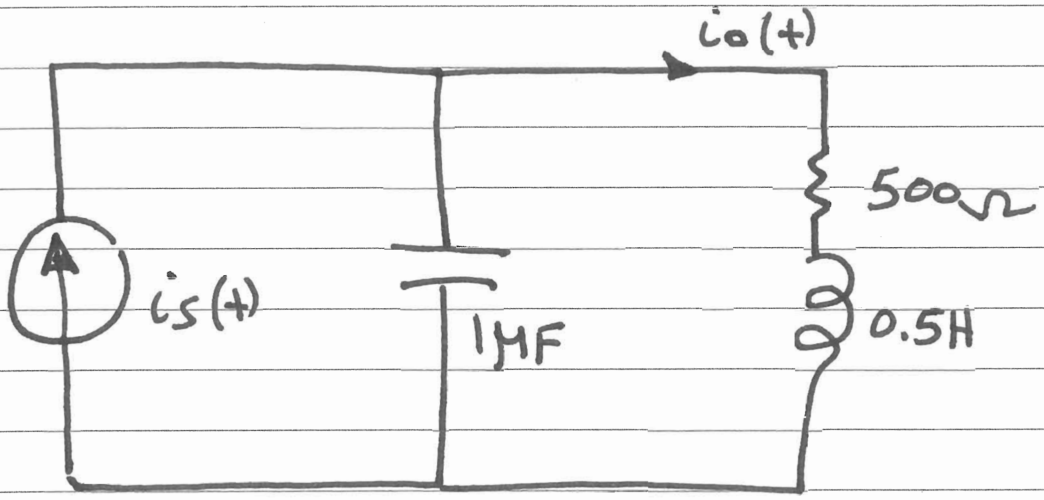


$$\vec{V}_o = \frac{-j2}{-j2 + 10 \parallel j5} \cdot 10\angle 75^\circ$$

$$\vec{V}_o = 7.071\angle -60^\circ\ \text{V}$$

$$\therefore v_o(t) = 7.071\ \text{Cos}\left(10t - 60^\circ\right)\ \text{V}$$

Calculate $i_o(t)$

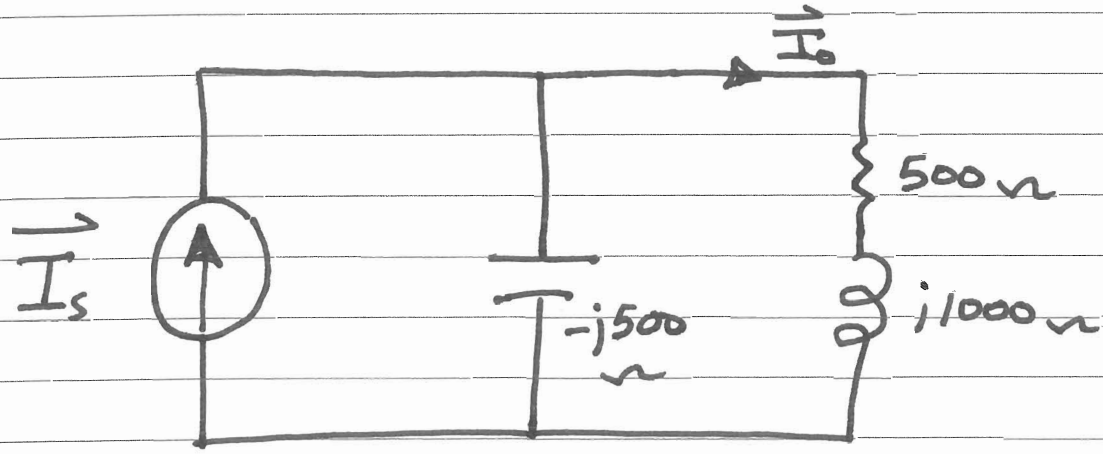


$$i_s(t) = 0.05 \cos 2000t \quad A$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j500\ \Omega$$

$$Z_L(j\omega) = j\omega L = j1000\ \Omega$$

$$\vec{I_s} = 0.05\angle 0^\circ \quad A$$

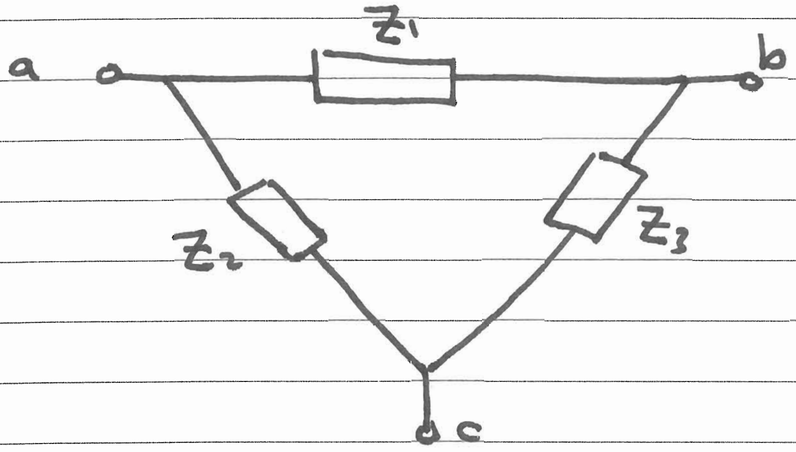


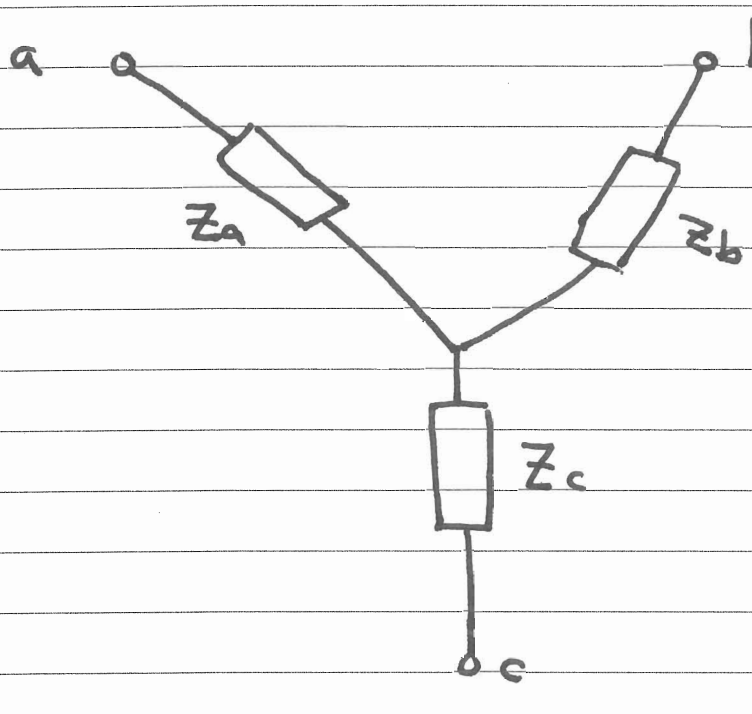
$$\vec{I_o} = \frac{-j500}{-j500 + 500 + j1000}\left(0.05\angle 0^\circ\right)$$

$$\vec{I_o} = 0.03535 \angle -45^\circ \text{ A}$$

$$\therefore i_o(t) = 0.03535 \cos(2000t - 45^\circ) \text{ A}$$

Y-Δ Transformation





$$Z_a = \frac{Z_1 Z_2}{Z_1 + Z_2 + Z_3}$$

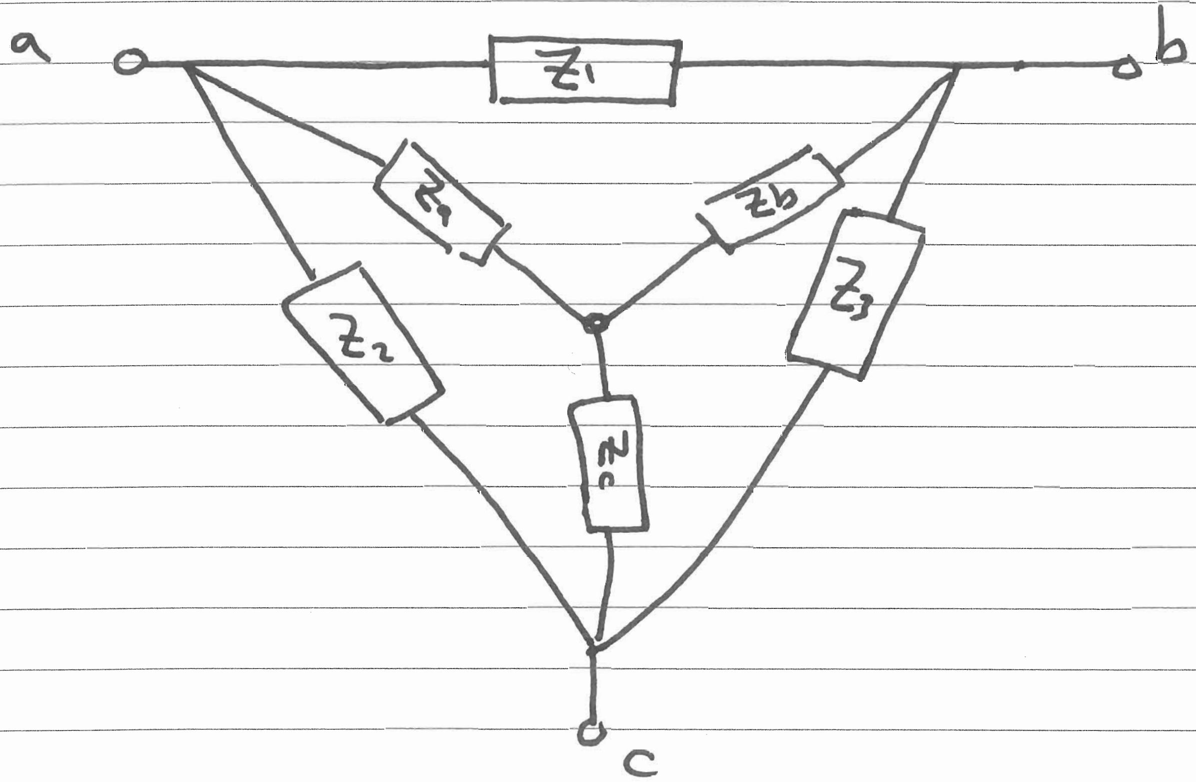
$$Z_b = \frac{Z_1 Z_3}{Z_1 + Z_2 + Z_3}$$

$$Z_c = \frac{Z_2 Z_3}{Z_1 + Z_2 + Z_3}$$

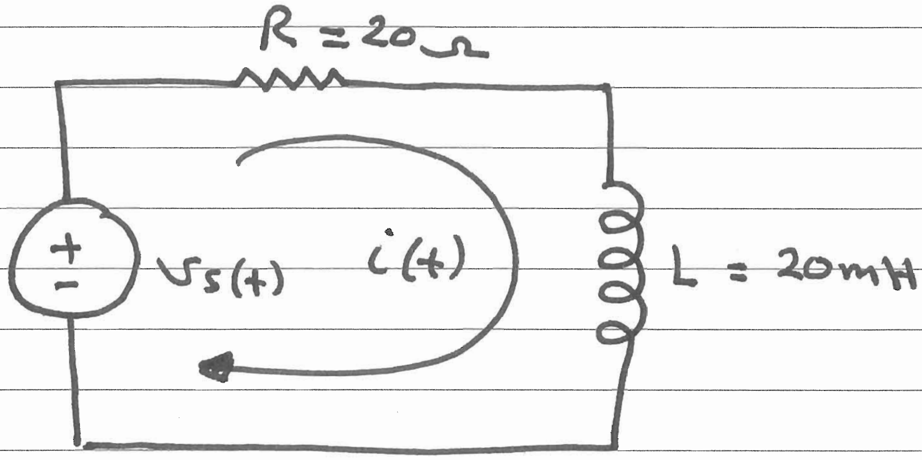
$$Z_1 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_c}$$

$$Z_2 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_b}$$

$$Z_3 = \frac{Z_a Z_b + Z_b Z_c + Z_c Z_a}{Z_a}$$



Series RL circuit



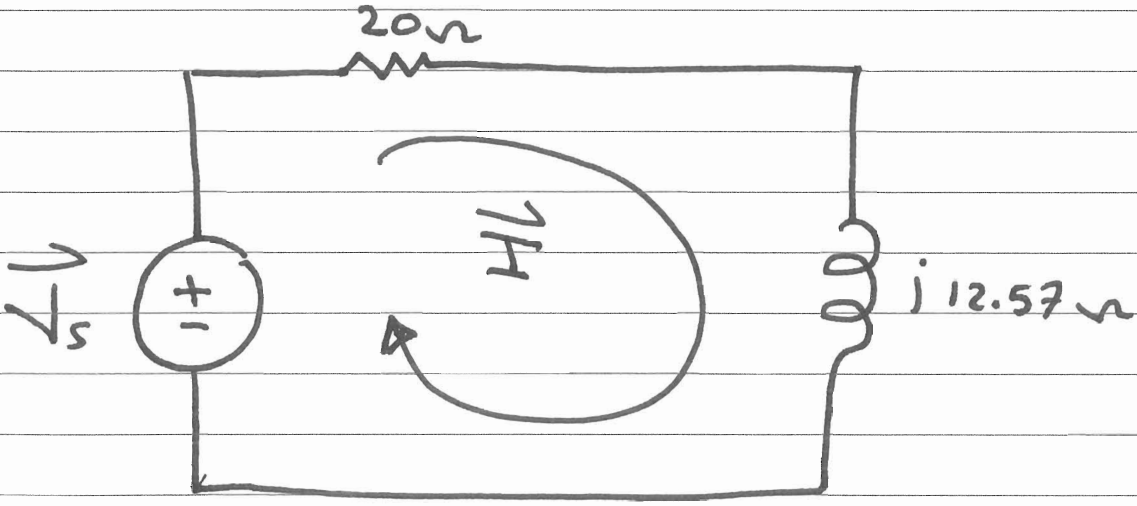
$v_s(t) = 60 \cos(200\pi t)$ V

Find $i(t)$

$Z_R(j\omega) = 20\ \Omega$

$Z_L(j\omega) = j12.57\ \Omega$

$\vec{V}_s = 60\angle 0^\circ$ V



KVL : $\vec{V}_s = \vec{V}_R + \vec{V}_L$

$60\angle 0^\circ = 20\vec{I} + j12.57\vec{I}$

$$\vec{I} = \frac{60\angle 0^\circ}{20 + j12.57} = \frac{60\angle 0^\circ}{23.6\angle 32.1^\circ}$$

$$\therefore \quad \vec{I} = 2.54\angle -32.1^\circ \quad A$$

$$\vec{V_R} = 20\vec{I} = 50.8\angle -32.1^\circ \quad V$$

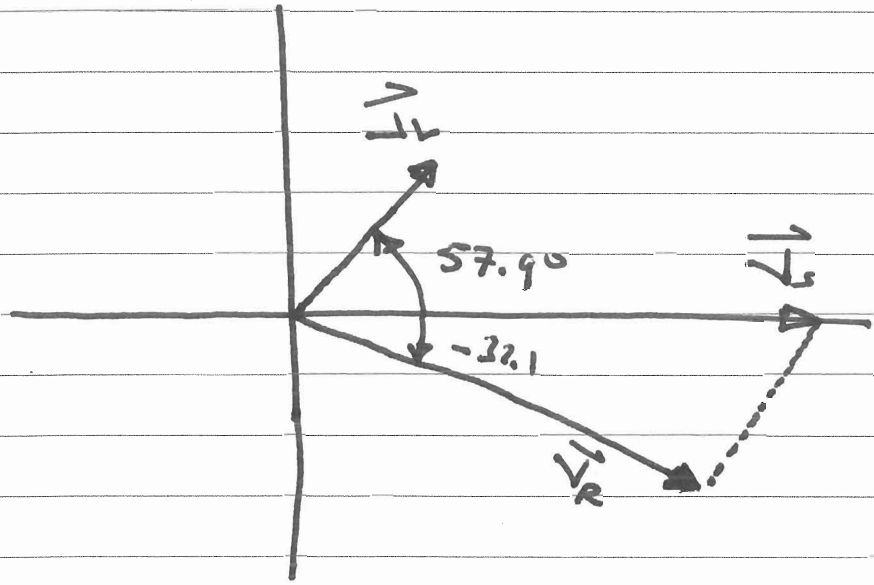
$$\vec{V_L} = j12.57\,\vec{I} = 31.9\angle +57.9^\circ \quad V$$

$\vec{V_L}$ Leads $\vec{V_R}$ by 90°

$\vec{I_L}$ Lags $\vec{V_s}$ by 32.1°

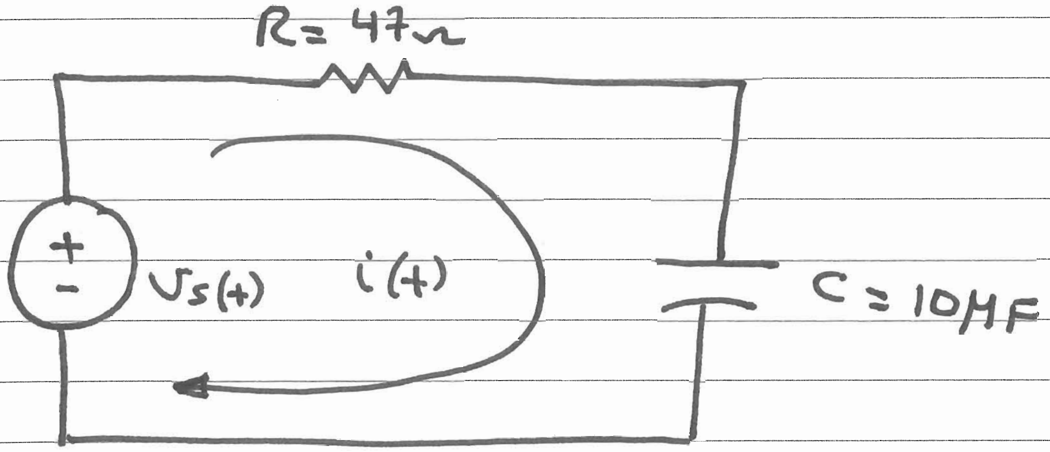
$$Z_{eq} = 20 + j12.57 \ \Omega \quad \text{inductive}$$

$$= 23.6\angle 32.1^\circ \ \Omega \quad \text{inductive}$$



phasor diagram

Series RC Circuit

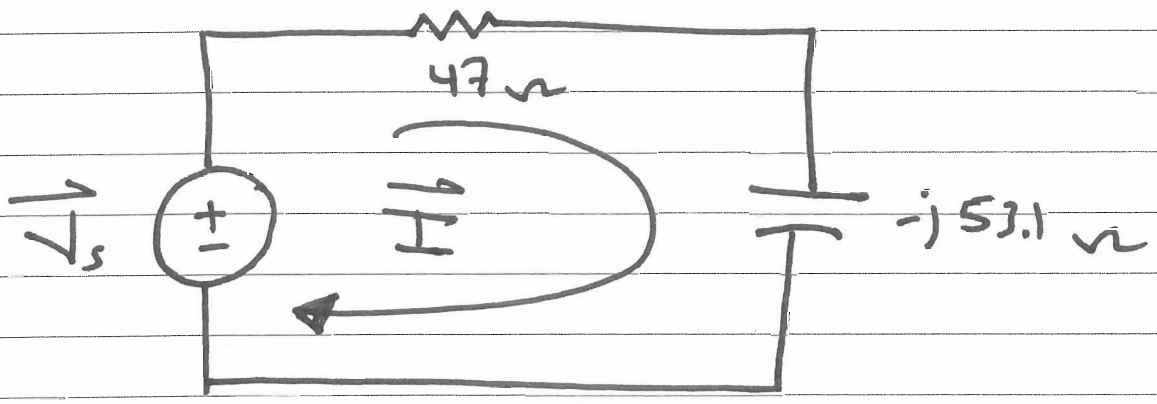


$$v_s(t) = 100 \cos 600\pi t \ \underline{V}$$

$$Z_R(j\omega) = 47\ \Omega$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j\,53.1\ \Omega$$

$$\vec{V}_s = 100\,\underline{\angle 0^\circ} \quad V$$



KVL :

$$\vec{V}_s = 47\,\vec{I} - j\,53.1\,\vec{I}$$

$$\vec{I} = \frac{\vec{V}_s}{47 - j\,53.1} = \frac{100\angle 0^\circ}{47 - j\,53.1}$$

$$\vec{I} = \frac{100\angle 0^\circ}{70.9\angle -48.5^\circ}$$

$$\vec{I} = 1.41\angle 48.5^\circ \text{ A}$$

$\vec{I}$ Leads $\vec{V_s}$ by 48.5°

$\longrightarrow$ Capacitive Circuit

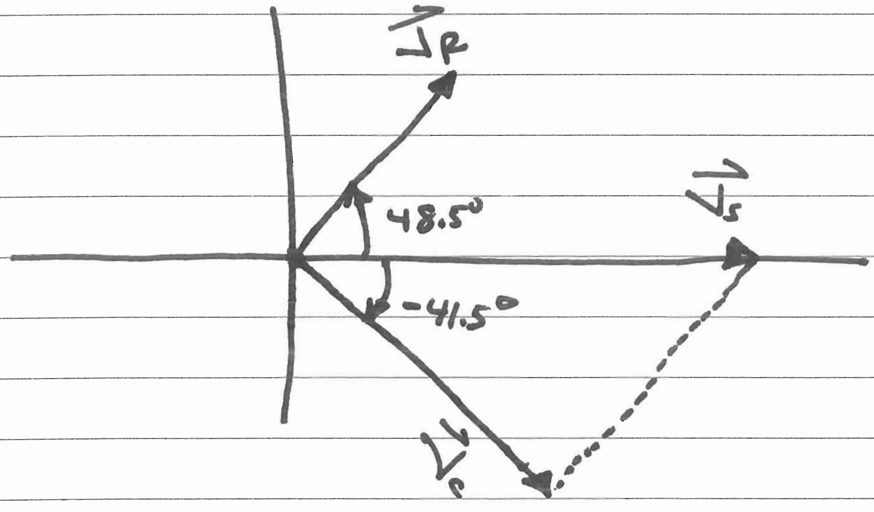
$Z(j\omega) = 47 - j\,53.1^\circ$ Capacitive

$Z(j\omega) = 70.9\angle -48.5^\circ$ Capacitive

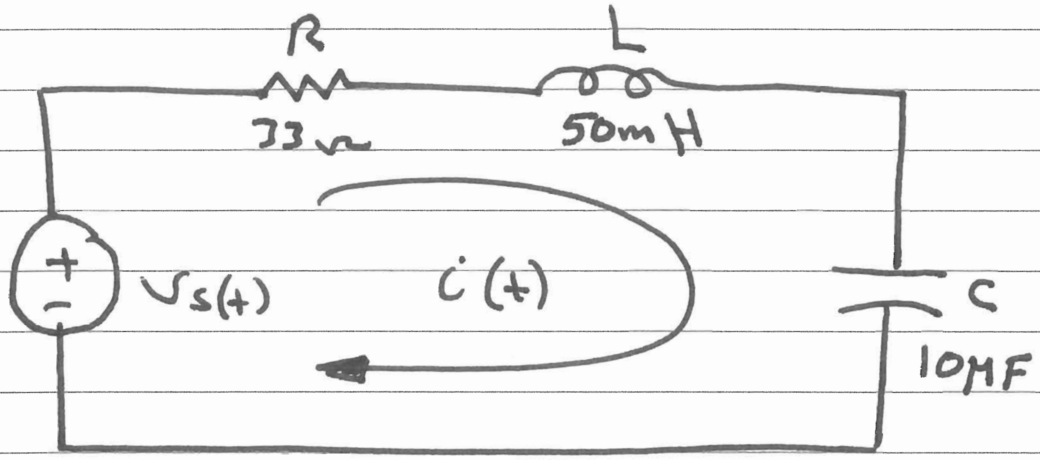
$$\vec{V_R} = 47\vec{I} = 66.3\angle 48.5^\circ \ \underline{V}$$

$$\vec{V_c} = -j\,53.1\,\vec{I} = 74.9\angle -41.5^\circ \ \underline{V}$$

$\vec{V_c}$ Lags $\vec{I}$ by 90°



Series RLC

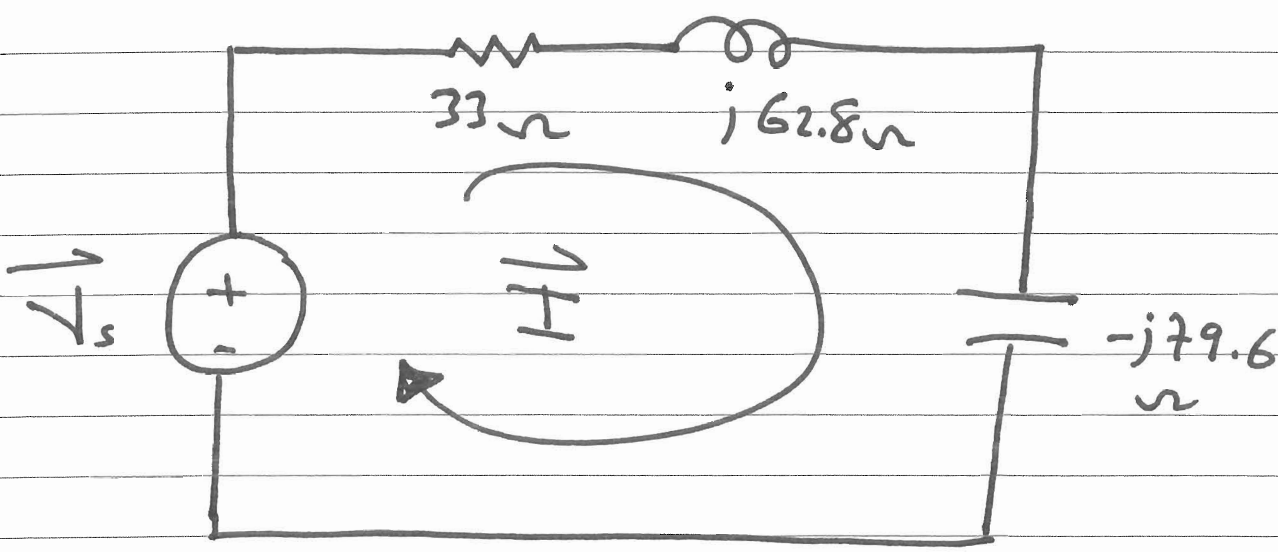


$$v_s(t) = 75 \cos 400\pi t \ \underline{V}$$

$$Z_R(j\omega) = 33\ \Omega$$

$$Z_L(j\omega) = j\omega L = j\,62.8\ \Omega$$

$$Z_C(j\omega) = -j\frac{1}{\omega C} = -j\,79.6\ \Omega$$



KVL: $\vec{V}_s = \vec{V}_R + \vec{V}_L + \vec{V}_C$

$$\vec{V_s} = 33\vec{I} + j62.8\vec{I} - j79.6\vec{I}$$

$$\vec{I} = \frac{\vec{V_s}}{33 - j16.8} = \frac{75\angle 0^\circ}{37\angle -27^\circ}$$

$$\vec{I} = 2.03\angle 27^\circ \quad A$$

$\vec{I}$ Leads $\vec{V_s}$ by 27°

$\therefore$ Capacitive Circuit

$$Z_{eq} = R + j\omega L - j\frac{1}{\omega C}$$

$$Z_{eq} = 33 + j62.8 - 79.6$$

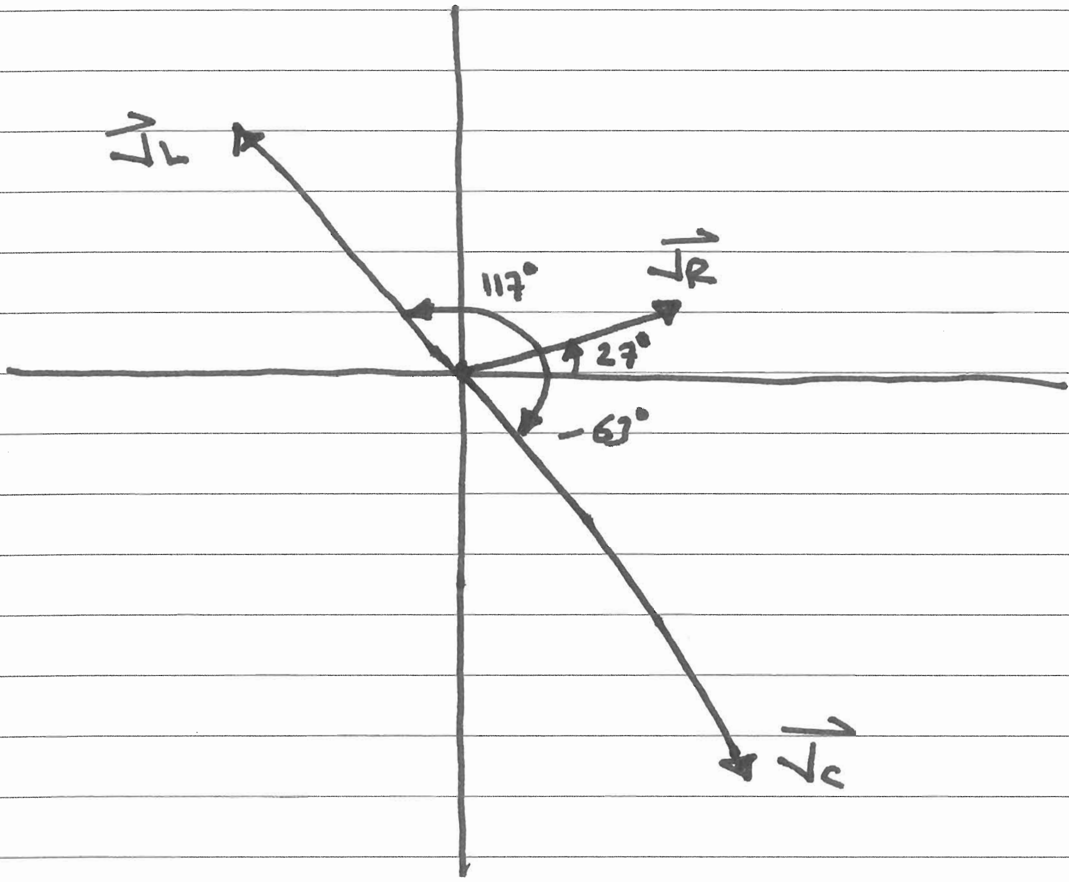
$$Z_{eq} = 33 - j16.8 \quad \Omega \qquad \text{Capacitive}$$

$$Z_{eq} = 37\angle -27^\circ \quad \Omega \qquad \text{Capacitive}$$

$$\vec{V_R} = R\vec{I} = 67\angle 27^\circ \quad V$$

$$\vec{V_L} = j\omega L\vec{I} = 127\angle 117^\circ \quad V$$

$$\vec{V_C} = -j\frac{1}{\omega C}\vec{I} = 162\angle -63^\circ \quad V$$

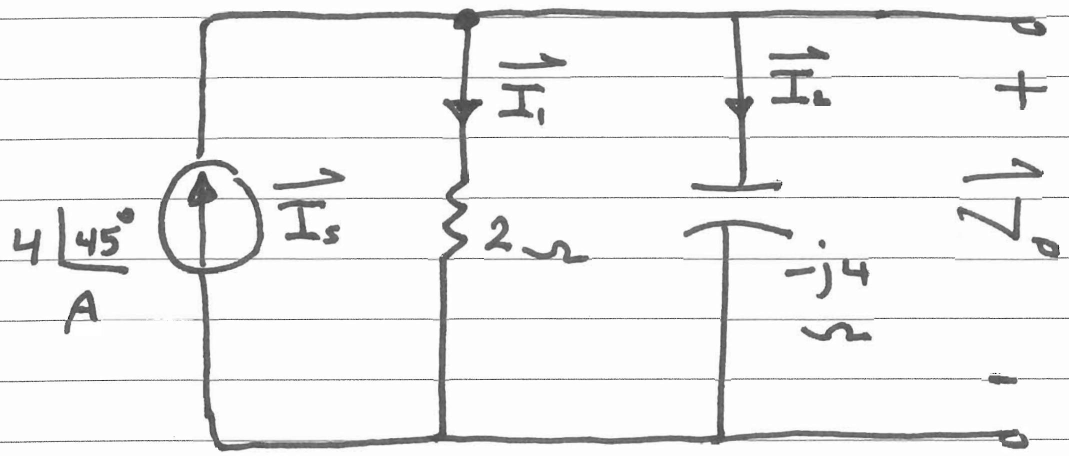


If $\quad j\omega L - j\dfrac{1}{\omega C} = 0$

$$\omega = \frac{1}{\sqrt{LC}}$$

resonant frequency

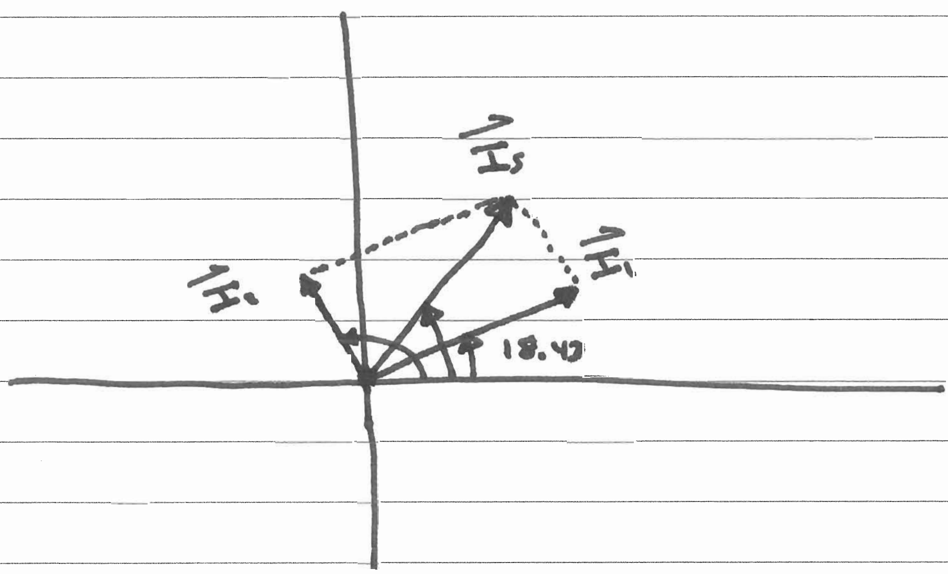
$Z_{eq} = R$ resistive



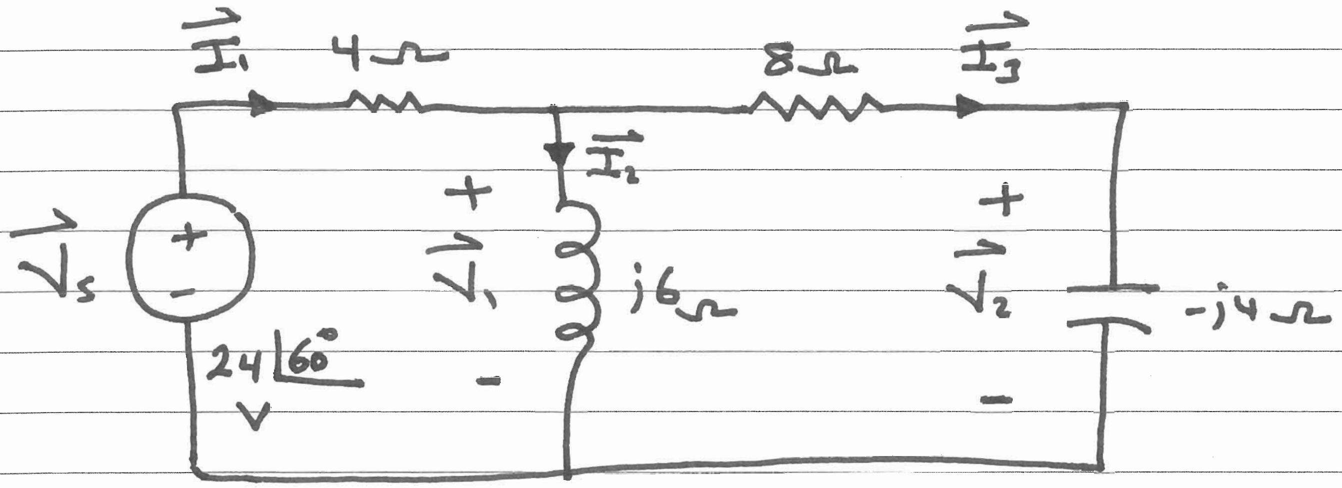
$$\vec{I}_1 = \frac{-j4}{-j4+2}\vec{I}_s = 3.578\angle 18.435^\circ \text{ A}$$

$$\vec{I}_2 = \frac{2}{2-j4}\vec{I}_s = 1.789\angle 108.435^\circ \text{ A}$$

$$\vec{V}_0 = 2\vec{I}_1 = 7.156\angle 18.435^\circ \text{ V}$$



phasor diagram



Calculate all the voltages and currents

$$Z_{eq} = 4 + j6 \,\|\, (8 - j4)$$

$$Z_{eq} = 9.604 \angle 30.964^\circ \ \Omega$$

$$\vec{I_1} = \frac{\vec{V_s}}{Z_{eq}} = \frac{24 \angle 60^\circ}{9.604 \angle 30.964^\circ} = 2.498 \angle 29.036^\circ \ A$$

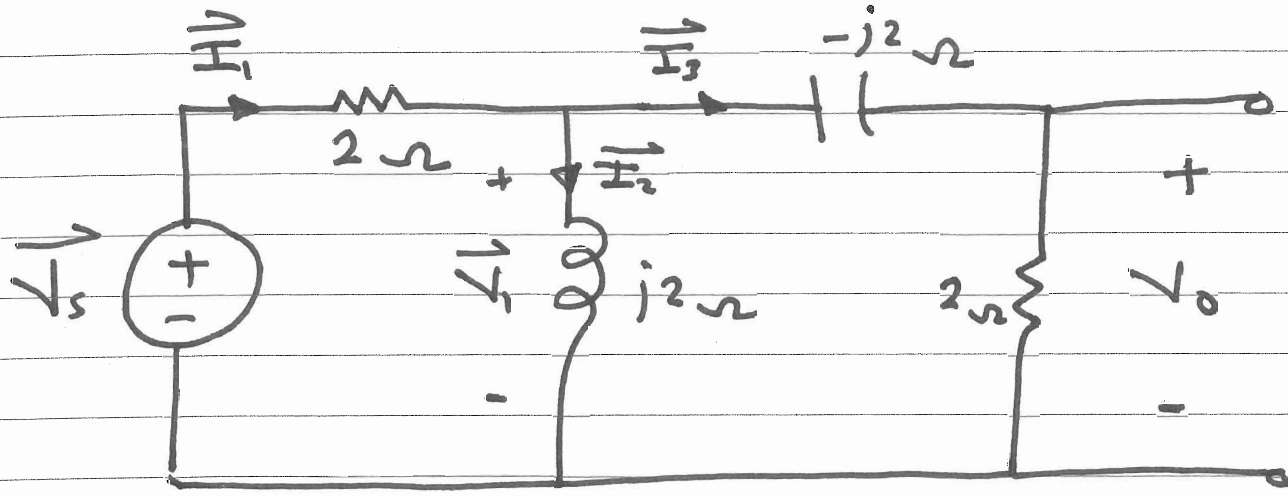
$$\vec{I_3} = \frac{j6}{j6 + 8 - j4} \vec{I_1} = 1.82 \angle 105^\circ \ A$$

$$\vec{I_2} = \frac{8 - j4}{8 - j4 + j6} \vec{I_1} = 2.71 \angle -11.58 \ A$$

$$\vec{V_1} = j6 \vec{I_2} = 16.26 \angle 78.42^\circ \ V$$

$$\vec{V_2} = -j4 \vec{I_3} = 7.28 \angle 15^\circ \ V$$

If $\vec{V}_o = 8\angle 45^\circ$ V, find $\vec{V}_s$



$$I_3 = \frac{\vec{V}_o}{2} = 4\angle 45^\circ \text{ A}$$

$$\vec{V}_1 = (2 - j2)\vec{I}_3 = 11.314\angle 0^\circ$$

$$\vec{I}_2 = \frac{\vec{V}_1}{j2} = 5.657\angle -90^\circ \text{ A}$$

$$\vec{I}_1 = \vec{I}_2 + \vec{I}_3 = (2.828 - j2.829) \text{ A}$$

$$\vec{V}_s = 2\vec{I}_1 + \vec{V}_1$$

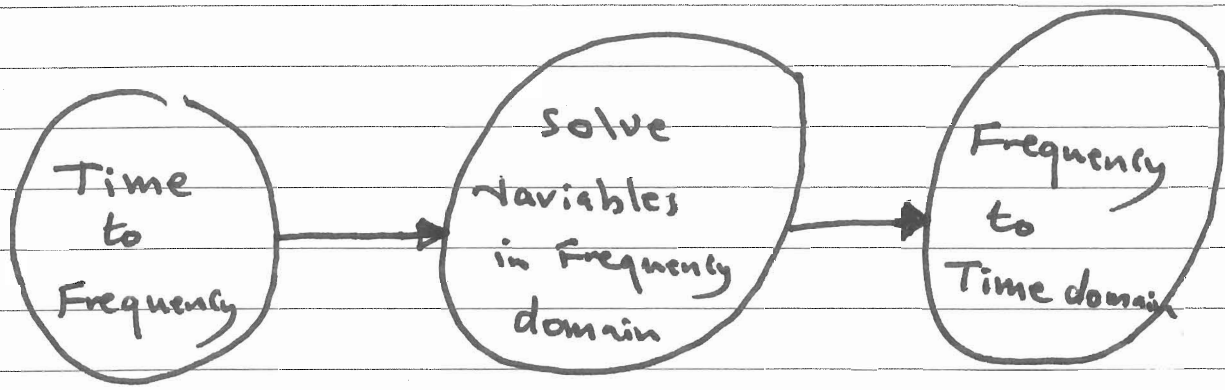
$$\vec{V}_s = 17.888\angle -18.439^\circ \quad \underline{V}$$

Steps to Analyze Ac Circuits

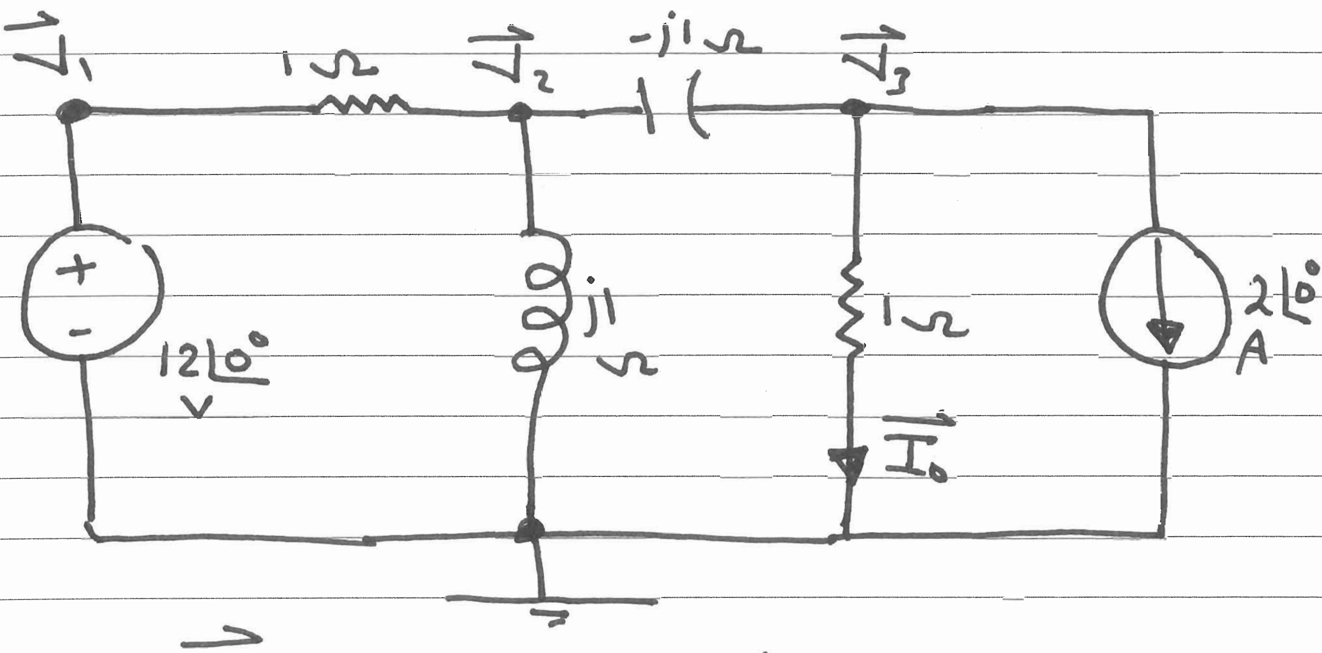
* Transform the Circuit to the phasor or frequency domain.

* Solve the problem using Circuit techniques (nodal analysis, mesh analysis, Superposition, etc .....)

* Transform the resulting phasor to the time domain.



Nodal Analysis



Find $\vec{I}_o$ using Nodal Analysis

$$\vec{I}_o = \frac{\vec{V}_3}{1}$$

$\vec{V}_1 = 12\angle 0^\circ$ Constrain equation

KCL at node 2 :

$$\frac{\vec{V}_2 - \vec{V}_1}{1} + \frac{\vec{V}_2}{j1} + \frac{\vec{V}_2 - \vec{V}_3}{-j1} = 0$$

$$-\vec{V}_1 + \vec{V}_2 - j\vec{V}_3 = 0$$

KCL at node 3 :

$$-2\angle 0^\circ = -\frac{1}{-j1}\vec{V_2} + \left(\frac{1}{-j1} + 1\right)\vec{V_3}$$

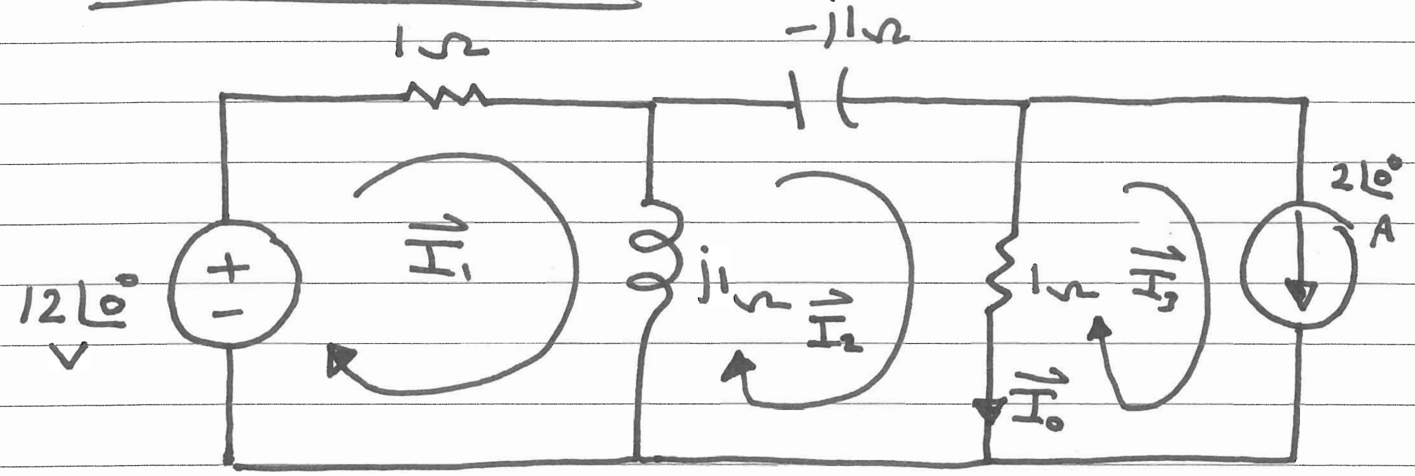
$$-2\angle 0^\circ = -j\vec{V_2} + (1+j)\vec{V_3}$$

Solving for $\vec{V_3}$;

$$\vec{V_3} = \left(\frac{8}{5} + j\frac{26}{5}\right) V$$

$$\therefore \vec{I_0} = \frac{\vec{V_3}}{1} = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

Mesh Analysis



Find $\vec{I_o}$ using Mesh Analysis

$$\vec{I_o} = \vec{I_2} - \vec{I_3}$$

KVL for mesh 1 :

$$12\angle 0^\circ = (1+j1)\vec{I_1} - j1\,\vec{I_2}$$

KVL for mesh 2 :

$$0 = -j1\,\vec{I_1} + (1+j1-j1)\vec{I_2} - \vec{I_3}$$

$$0 = -j1\,\vec{I_1} + \vec{I_2} - \vec{I_3}$$

$$\vec{I_3} = 2\angle 0^\circ \text{ A}$$ Constrain equation

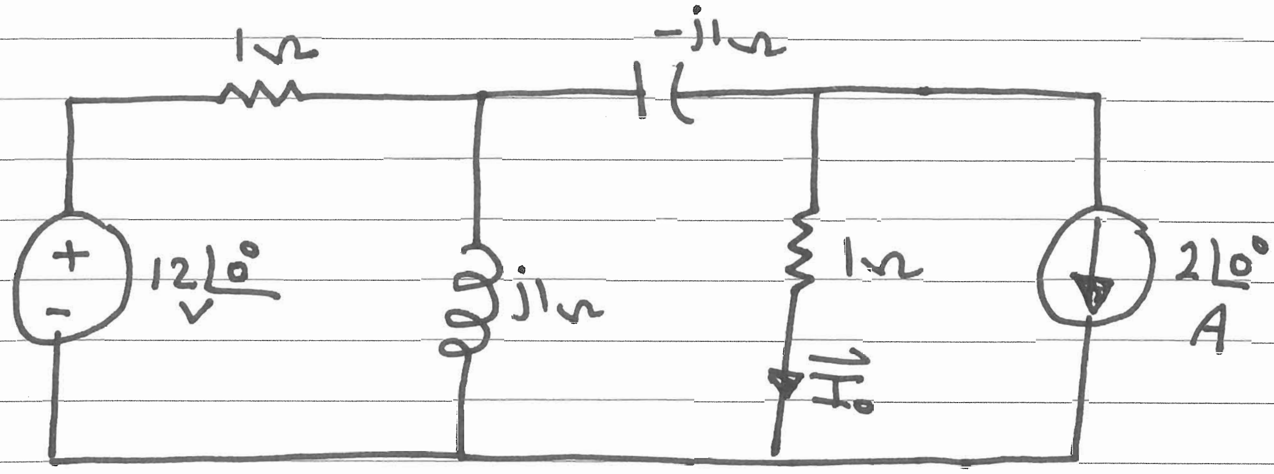
Solving for $\vec{I_2}$ and $\vec{I_3}$

$$\vec{I_2} = \left(\frac{18}{5} + j\,\frac{26}{5}\right) \text{A}$$

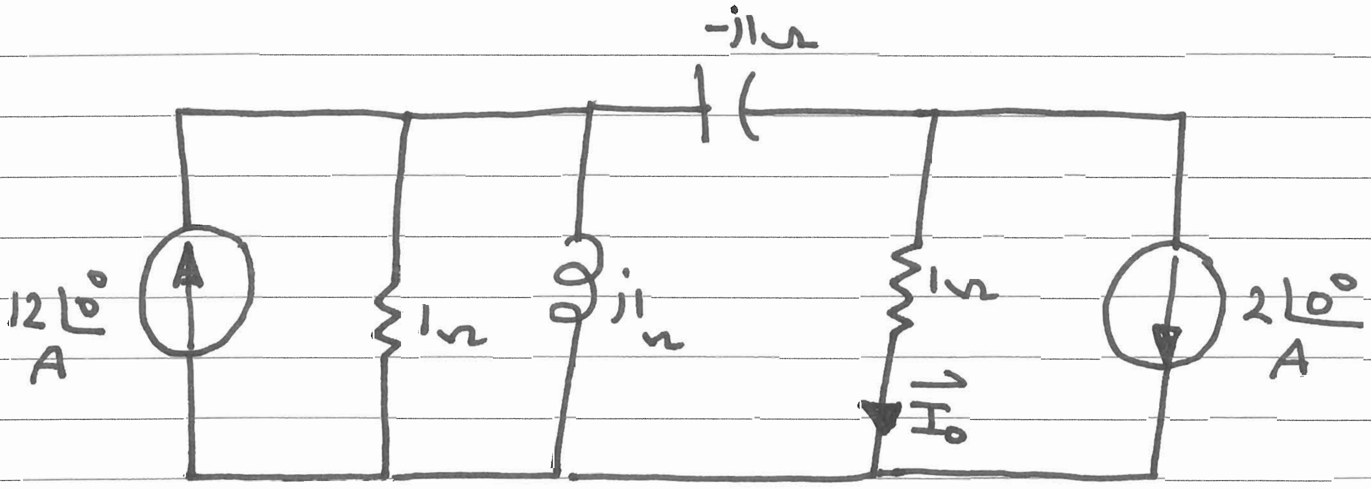
$$\vec{I_3} = 2\angle 0^\circ \text{ A}$$

$$\therefore \vec{I_o} = \left(\frac{8}{5} + j\frac{26}{5}\right) \text{ A}$$

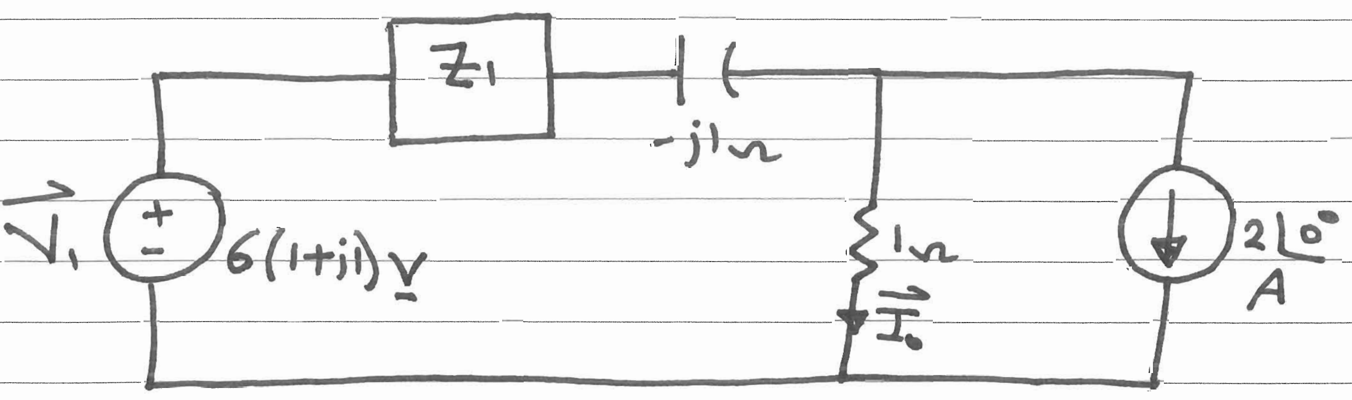
Source Transformation



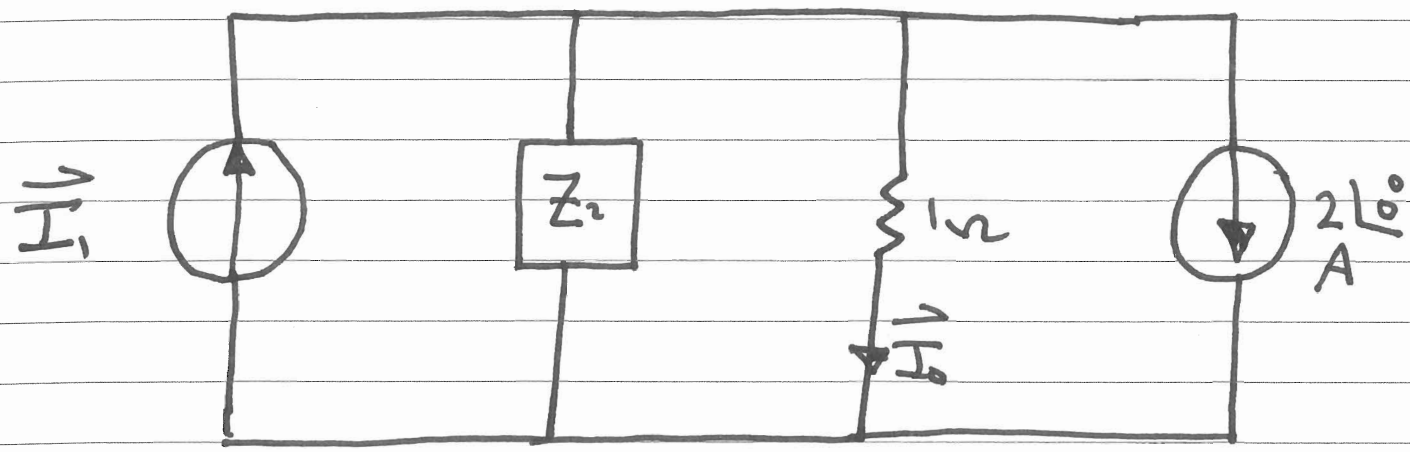
Find $\vec{I_o}$ using Source Transformation



$$Z_1 = 1\Omega \| j1\Omega = \left(\tfrac{1}{2} + j\tfrac{1}{2}\right)\Omega$$

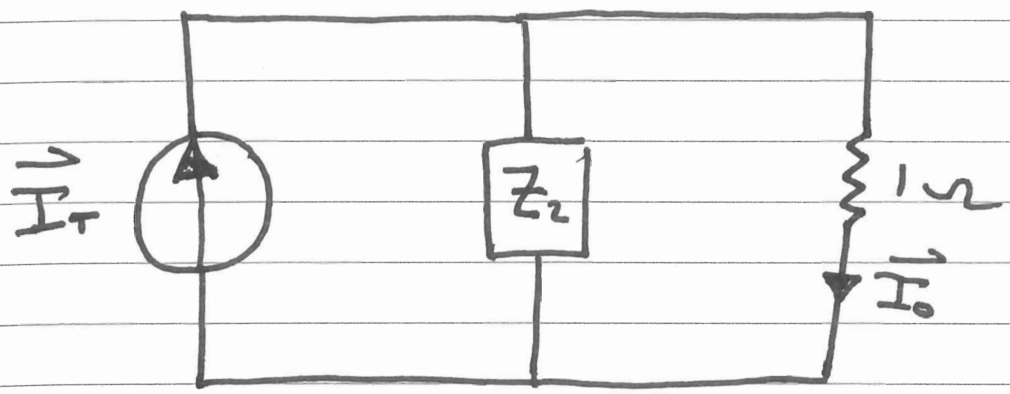


$$\vec{V_1} = 12\angle 0^\circ \cdot Z_1 = 6(1+j1)\ \underline{V}$$



$$\vec{I}_1 = \frac{\vec{V}_1}{Z_2} = \frac{12(1+j1)}{1-j1}$$

$$\dot{Z}_2 = -j1 + Z_1 = \left(\frac{1}{2} - j\frac{1}{2}\right)\Omega$$

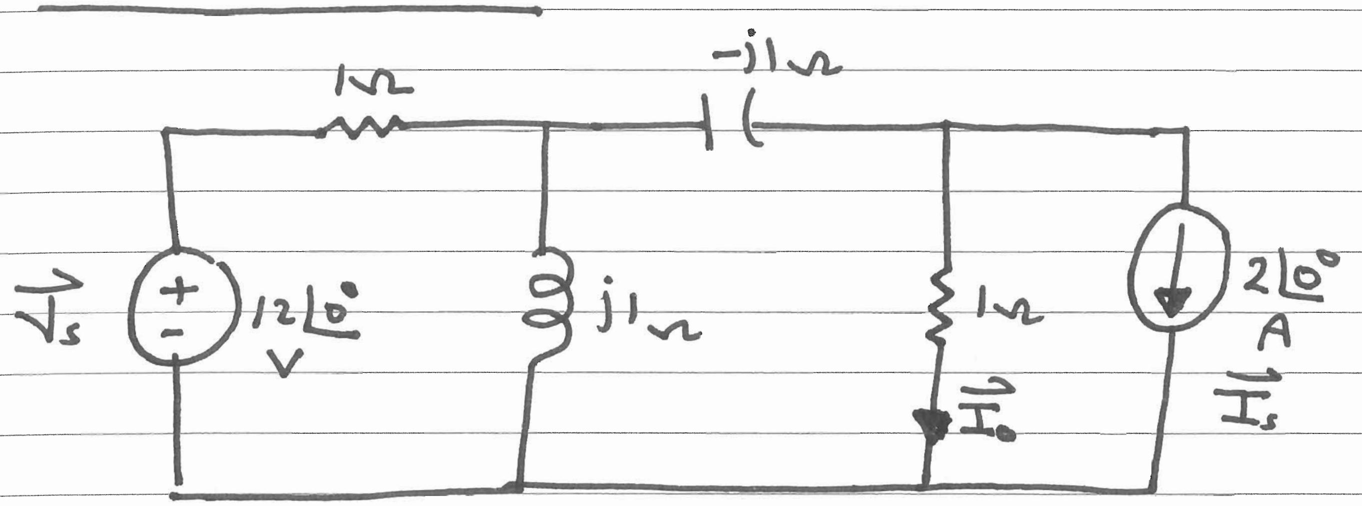


$$\vec{I}_T = \vec{I}_1 - 2\angle 0°$$

$$\vec{I}_T = \left(\frac{10+j14}{1-j1}\right) A$$

$$\vec{I}_o = \frac{Z_2}{Z_2+1}\,\vec{I}_T = \left(\frac{8}{5} + j\frac{26}{5}\right) A$$

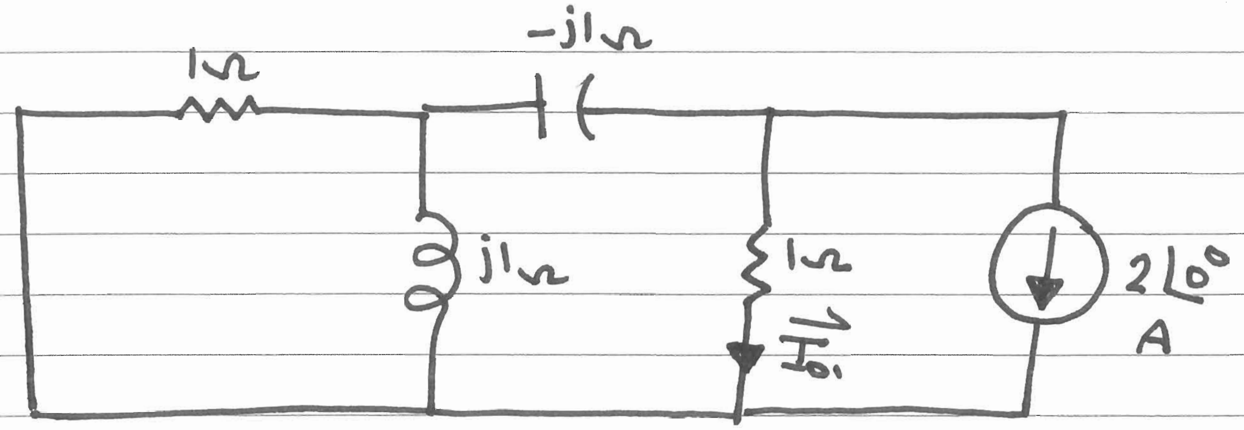
Superposition



Find $\vec{I_o}$ using Superposition

$$\vec{I_o} = \vec{I_{o1}} + \vec{I_{o2}}$$

1) let $\vec{V_s}$ OFF, and $\vec{I_s}$ on

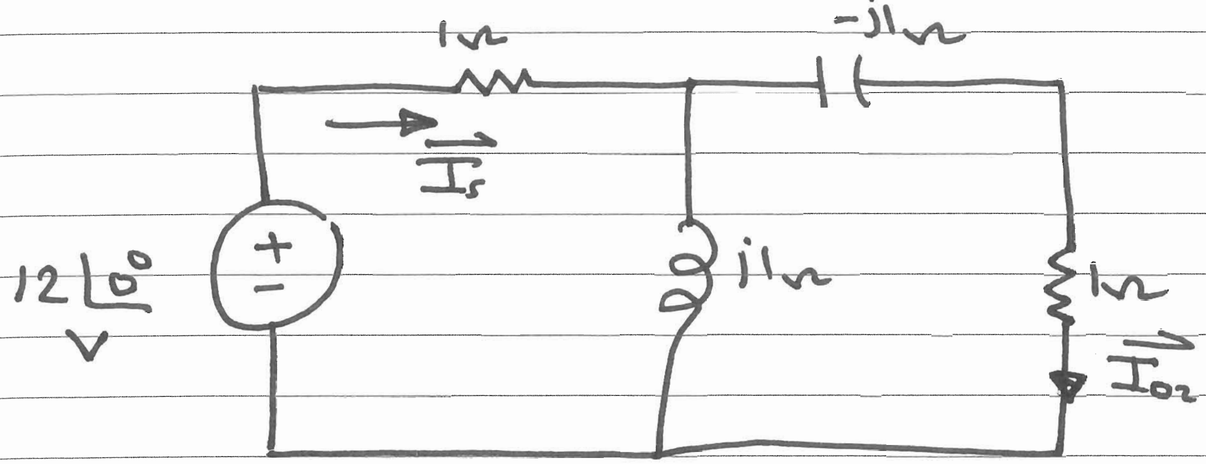


$$\vec{I_{o1}} = -2\angle 0^\circ \frac{Z_1}{Z_1 + 1}$$

$$Z_1 = -j1 + 1 \parallel j1 = -j1 + \frac{j1}{1+j1}$$

$$\vec{I_{o1}} = \frac{-2}{2+j1} \text{ A}$$

2) let $\vec{I_s}$ off, and $\vec{V_s}$ on



$$\vec{I_s} = \frac{12\angle 0^\circ}{Z_{eq}}$$

$$Z_{eq} = 1 + j1 \parallel (1 - j1) = (2 + j1)\,\Omega$$

$$\therefore \vec{I_s} = \frac{12\angle 0^\circ}{2 + j1} \text{ A}$$

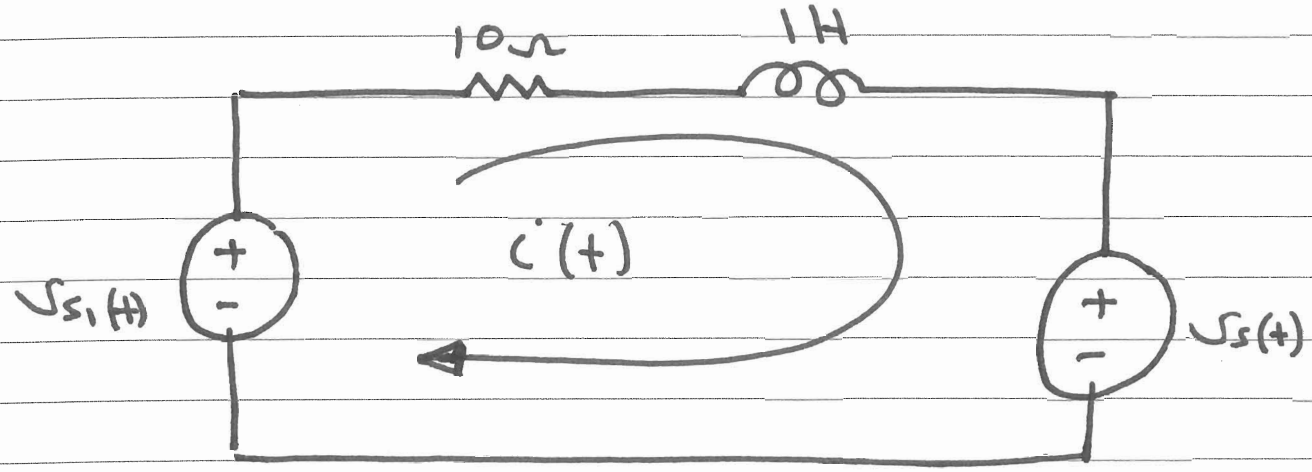
$$\vec{I_{o2}} = \vec{I_s}\,\frac{j1}{j1 + 1 - j1}$$

$$\vec{I_{o2}} = \vec{I_s} \cdot j1 = \frac{12}{1 - j2} \text{ A}$$

$$\therefore \vec{I_o} = \vec{I_{o1}} + \vec{I_{o2}}$$

$$\vec{I_o} = \left(\frac{8}{5} + j\,\frac{26}{5}\right) \text{A}$$

The Power of Superposition



$$v_{s1}(t) = 100 \cos 10t \ \underline{V}$$

$$v_{s2}(t) = 50 \cos(20t - 10^\circ) \ \underline{V}$$

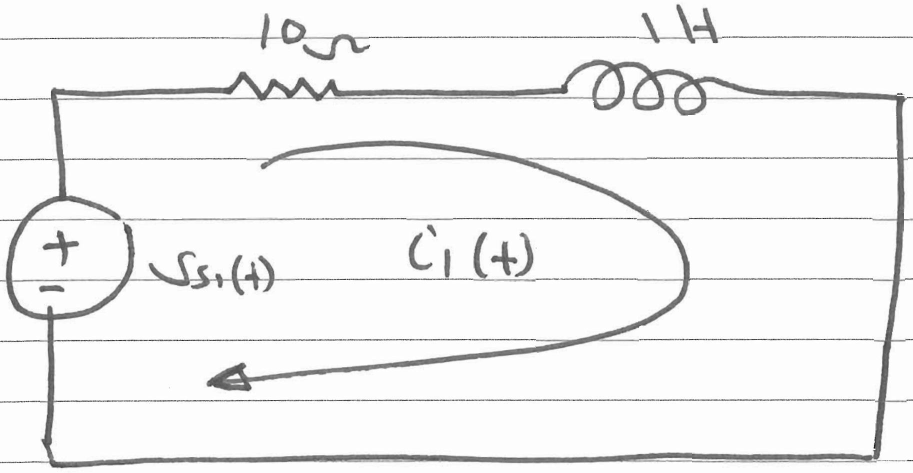
note that $\omega_1 = 10$ r/s and

$\omega_2 = 20$ r/s

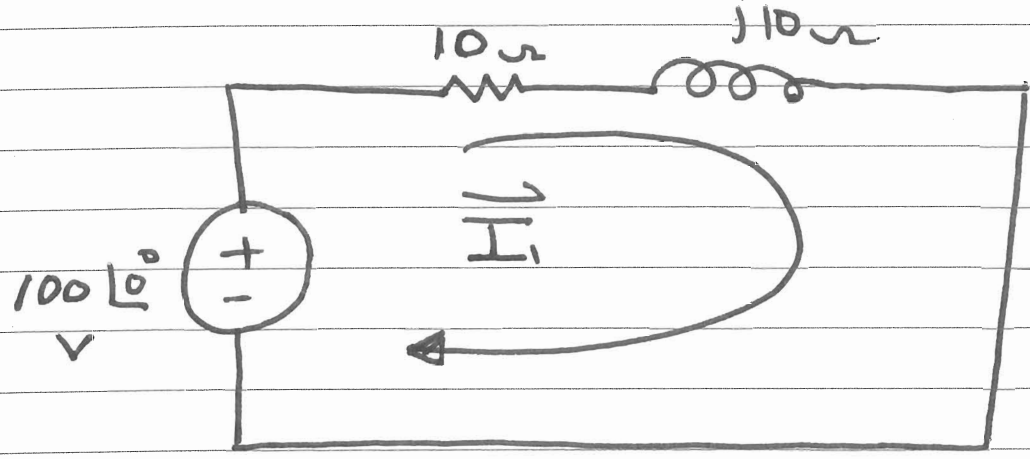
$\therefore$ Superposition is the Only method of analysis.

$$i(t) = i_1(t) + i_2(t)$$

1) Let $v_{s2}(t)$ OFF, and $v_{s1}(t)$ on



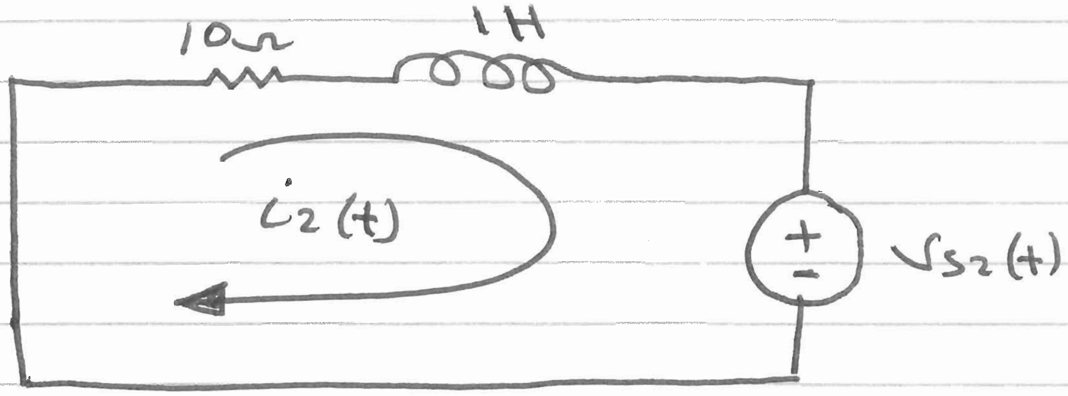
$v_{s1}(t) = 100 \cos 10t \ \underline{V}$



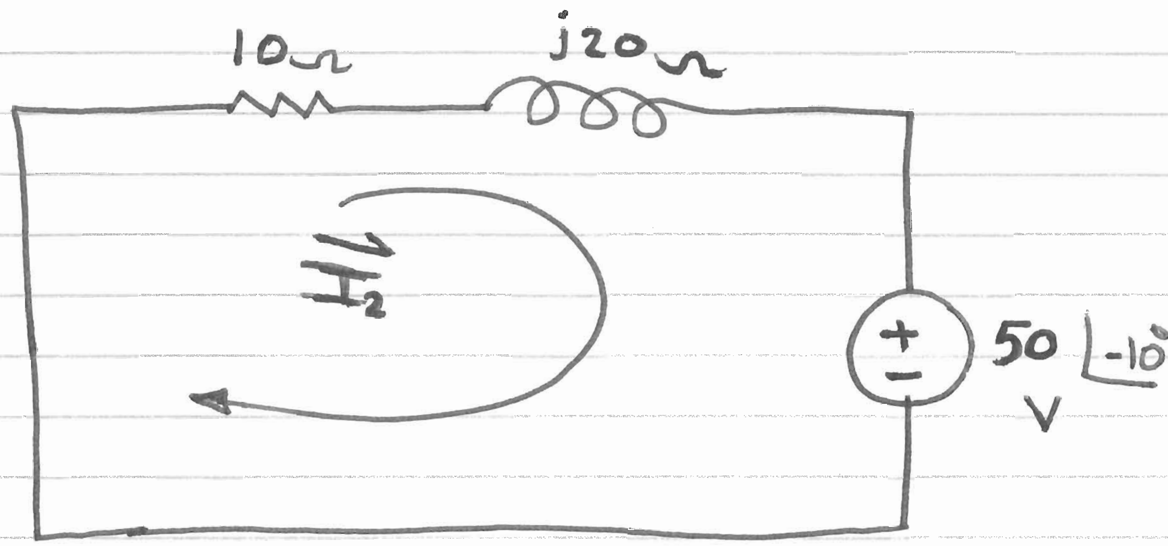
$$\vec{I}_1 = \frac{100\angle 0^\circ}{10 + j10} = 7.07\angle -45^\circ \ A$$

$$\therefore \ i_1(t) = 7.07 \cos(10t - 45^\circ) \ A$$

2) let $v_{s1}(t)$ OFF, and $v_{s2}(t)$ On



$$v_{s2}(t) = 50 \cos(20t - 10^\circ)\ V$$



$$\vec{I_2} = \frac{-50\angle -10^\circ}{10 + j20} = \frac{50\angle 170^\circ}{10 + j20}$$

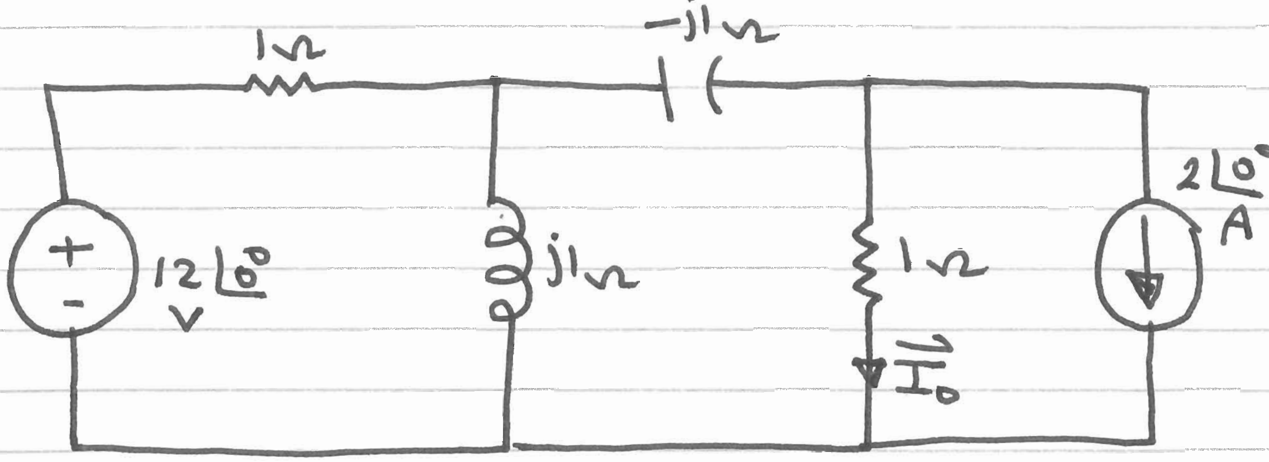
$$\vec{I_2} = 2.24\angle 106.57^\circ\ A$$

$$\therefore\ i_2(t) = 2.24 \cos(20t + 106.57^\circ)\ A$$

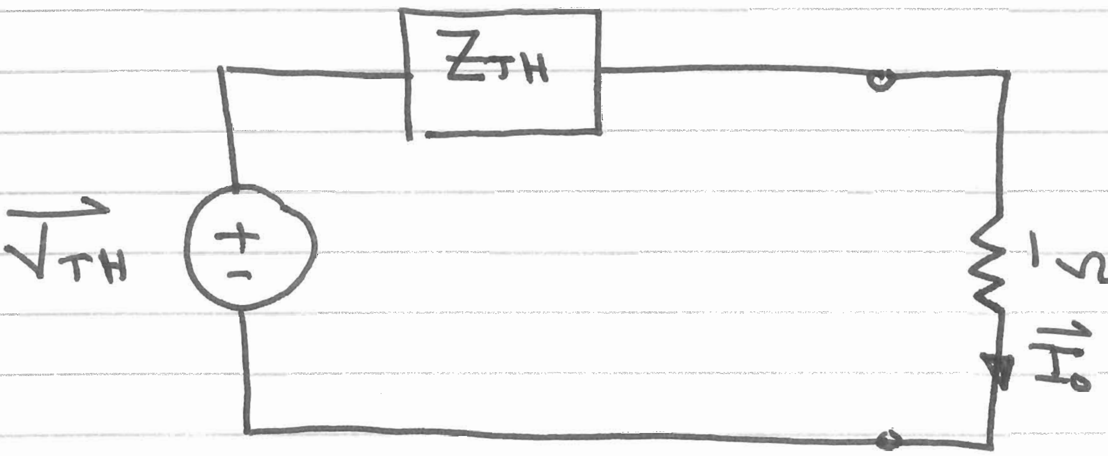
$$\therefore\ i(t) = i_1(t) + i_2(t)$$

$$i(t) = 7.07 \cos(10t - 45°)\ A$$
$$+ 2.24 \cos(20t + 106.57°)\ A$$

Thevenin's and Norton's Theorems

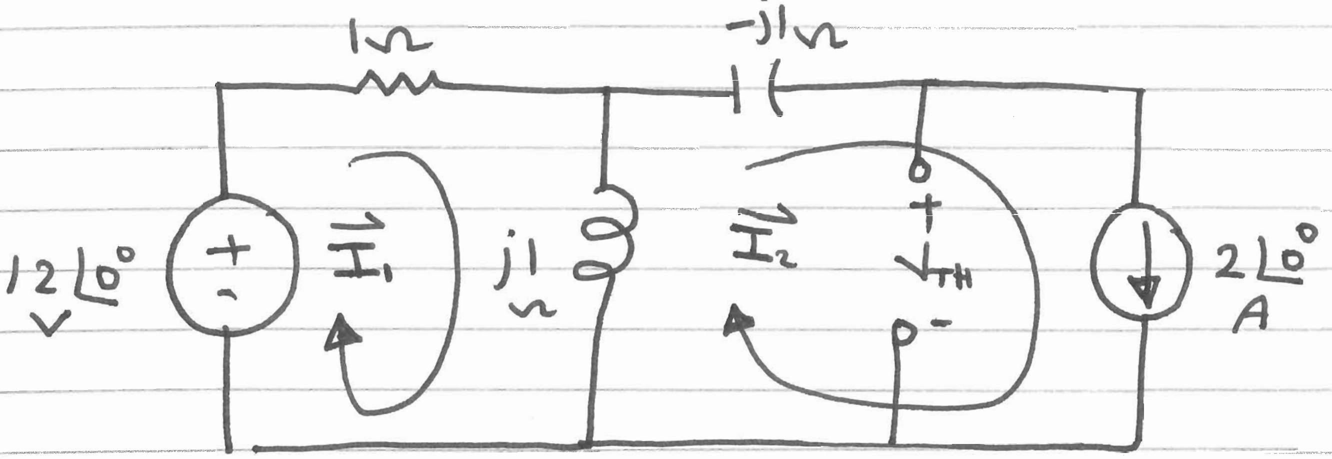


Find $\vec{I_o}$ using Thevenin's theorem



$$\vec{I_o} = \frac{\vec{V_{TH}}}{Z_{TH} + 1\Omega}$$

1) To find $\vec{V}_{TH}$



$$\vec{V}_{TH} = -(-j1\Omega)\,\vec{I}_2 + j1\Omega\,(\vec{I}_1 - \vec{I}_2)$$

$$\vec{I}_2 = 2\angle 0^\circ \quad \text{Constrain equation}$$

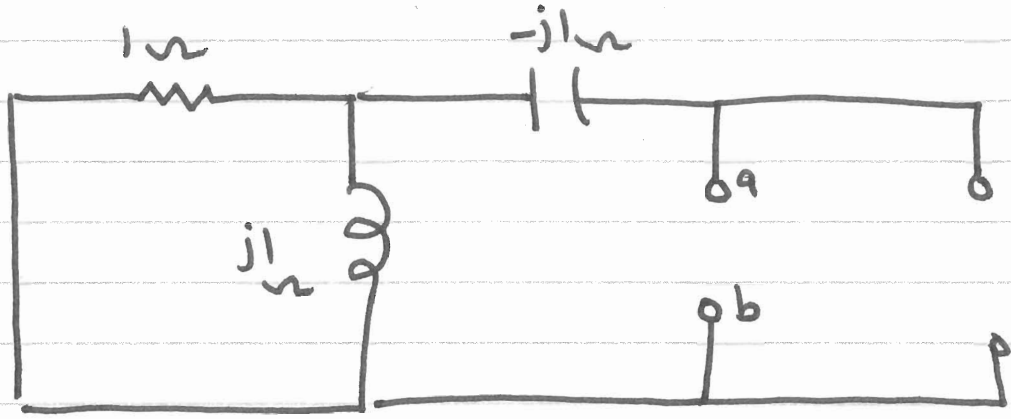
KVL for mesh 1 :

$$12\angle 0^\circ = (1+j1)\,\vec{I}_1 - j1\,\vec{I}_2$$

$$\therefore\ \vec{I}_1 = \left(\frac{12+j2}{1+j1}\right) A$$

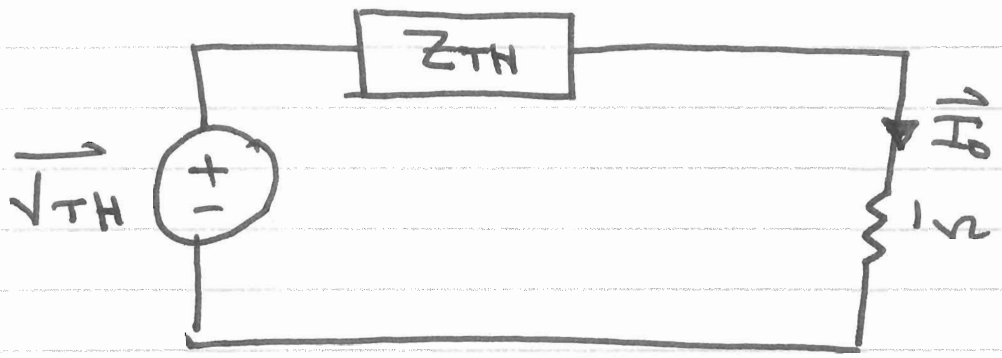
$$\therefore\ \vec{V}_{TH} = \left(\frac{-2+j12}{1+j1}\right) V$$

2) To find Z_{TH}, set all the independent sources to zero



$$Z_{TH} = -j1 + (1 \| j1)$$

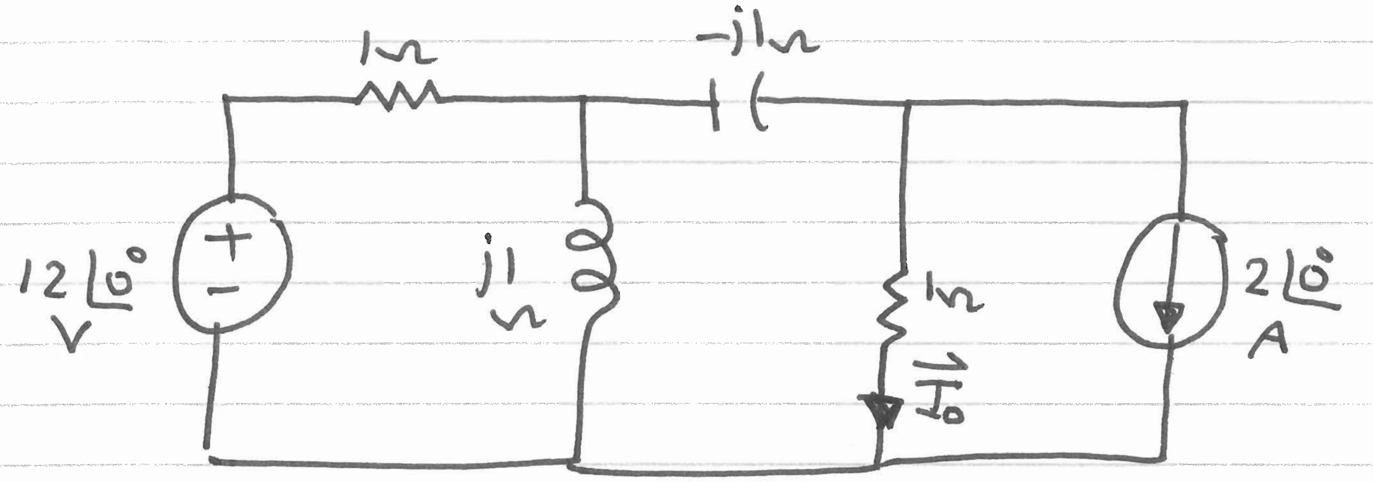
$$Z_{TH} = \left(\frac{1}{2} - j\frac{1}{2}\right) \Omega$$



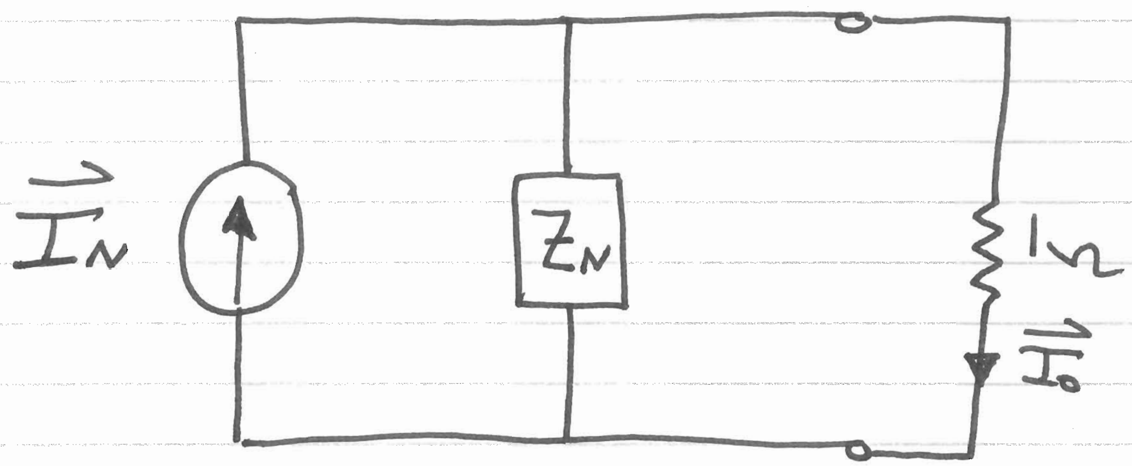
$$\vec{I_o} = \frac{\vec{V_{TH}}}{Z_{TH} + 1\Omega}$$

$$I_o = \left(\frac{8}{5} + j\,\frac{26}{5}\right) A$$

Norton's Theorem

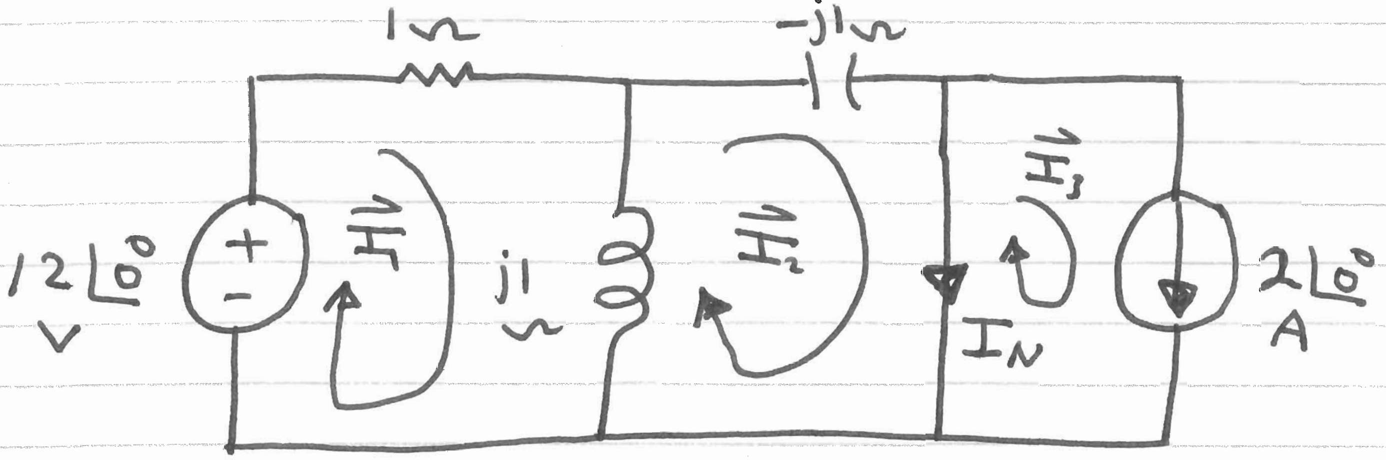


Find $\vec{I_o}$ using Norton's theorem



$$\vec{I_o} = \vec{I_N} \frac{Z_N}{Z_N + 1\Omega}$$

1) To find I_N



$$\vec{I_N} = \vec{I_2} - \vec{I_3}$$

$$\vec{I_3} = 2\angle 0^\circ \text{ A} \qquad \text{Constrain equation}$$

KVL for mesh 1 :

$$12\angle 0^\circ = (1+j1)\vec{I_1} - j1\vec{I_2}$$

KVL for mesh 2 :

$$0 = -j1\vec{I_1} + (j1-j1)\vec{I_2}$$

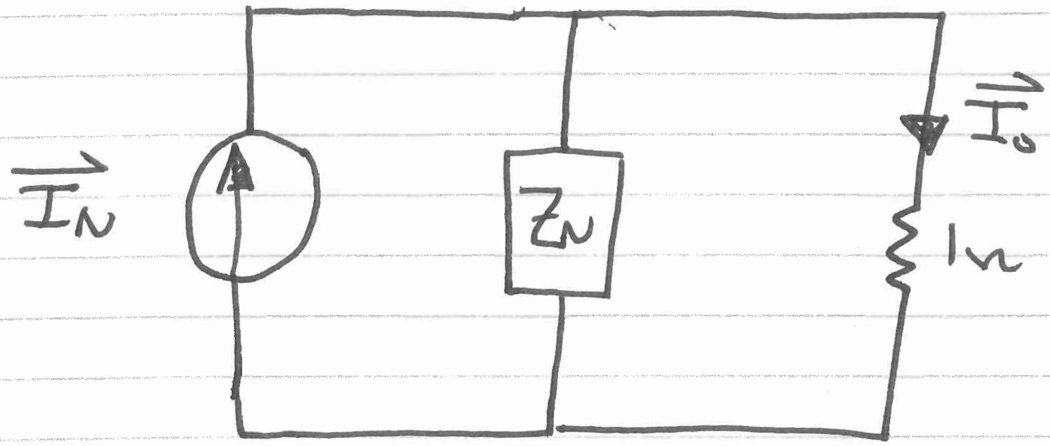
$$0 = -j1\vec{I_1}$$

$$\therefore \vec{I_1} = 0$$

$$\therefore \vec{I_2} = 12\angle 90^\circ \text{ A}$$

$$\therefore \vec{I_N} = \vec{I_2} - \vec{I_3} = -2 + j12$$

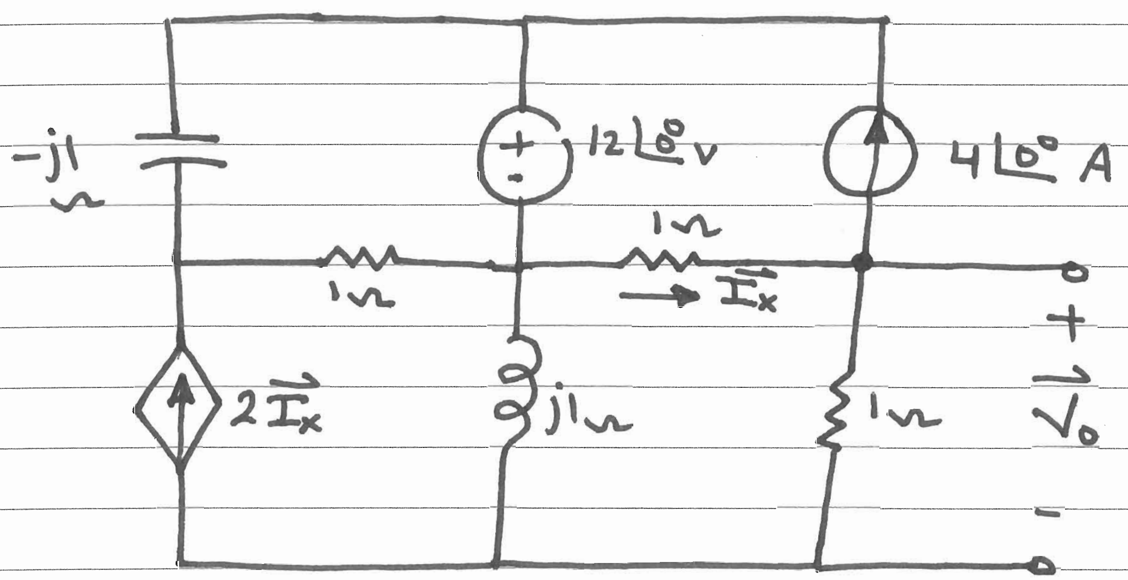
$$Z_N = Z_{TH} = \left(\frac{1}{2} - j\frac{1}{2}\right)\Omega$$



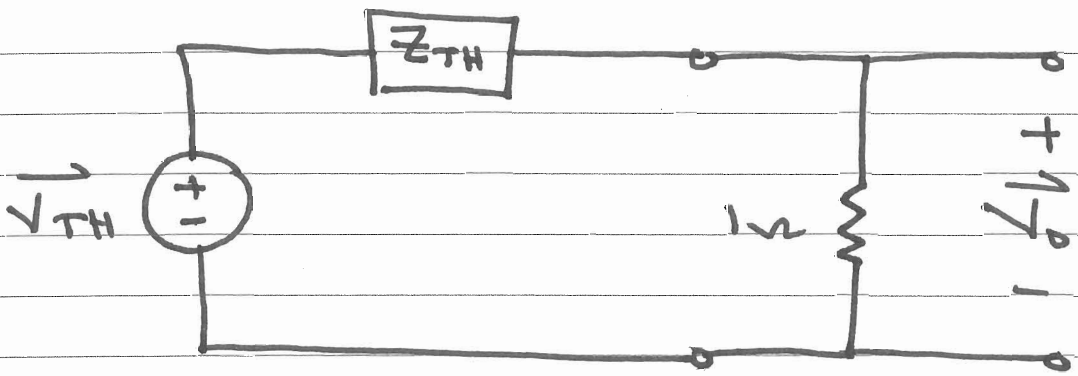
$$\vec{I_o} = \vec{I_N}\,\frac{Z_N}{Z_N + 1\Omega}$$

$$\vec{I_o} = \left(\frac{8}{5} + j\,\frac{26}{5}\right) A$$

Thevenin's Theorem

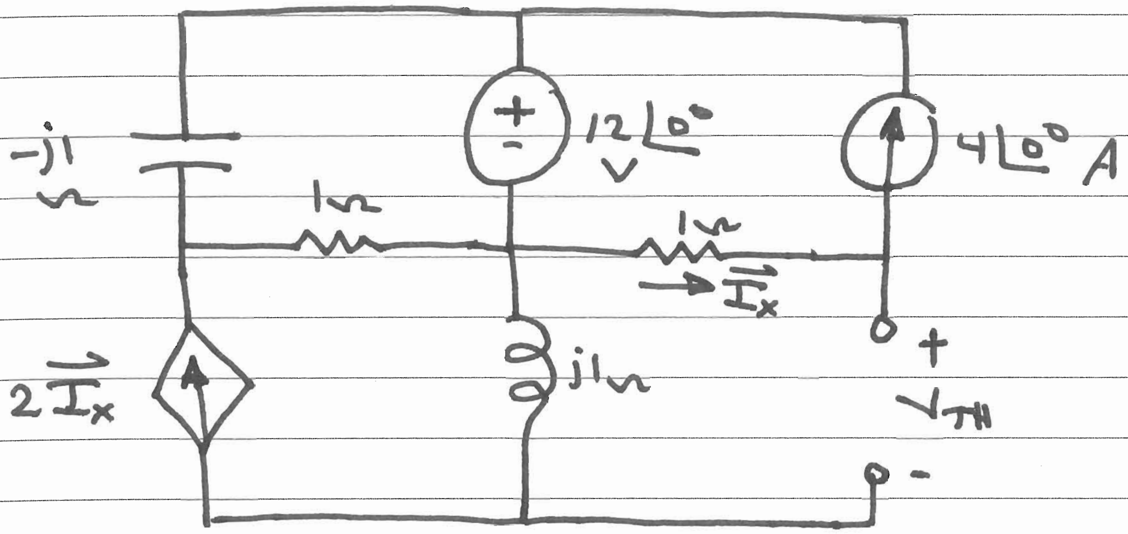


Find $\vec{V}_o$ using Thevenin's theorem



$$\vec{V}_o = \frac{1\Omega}{1\Omega + Z_{TH}}\,\vec{V}_{TH}$$

1) To find $\vec{V}_{TH}$



$$V_{TH} = -1\Omega\ \vec{I}_x + j1\Omega\,(2\vec{I}_x)$$

$$\vec{I}_x = 4\angle 0°\ A$$

$$\therefore\ \vec{V}_{TH} = (-4 + j8)\ V$$

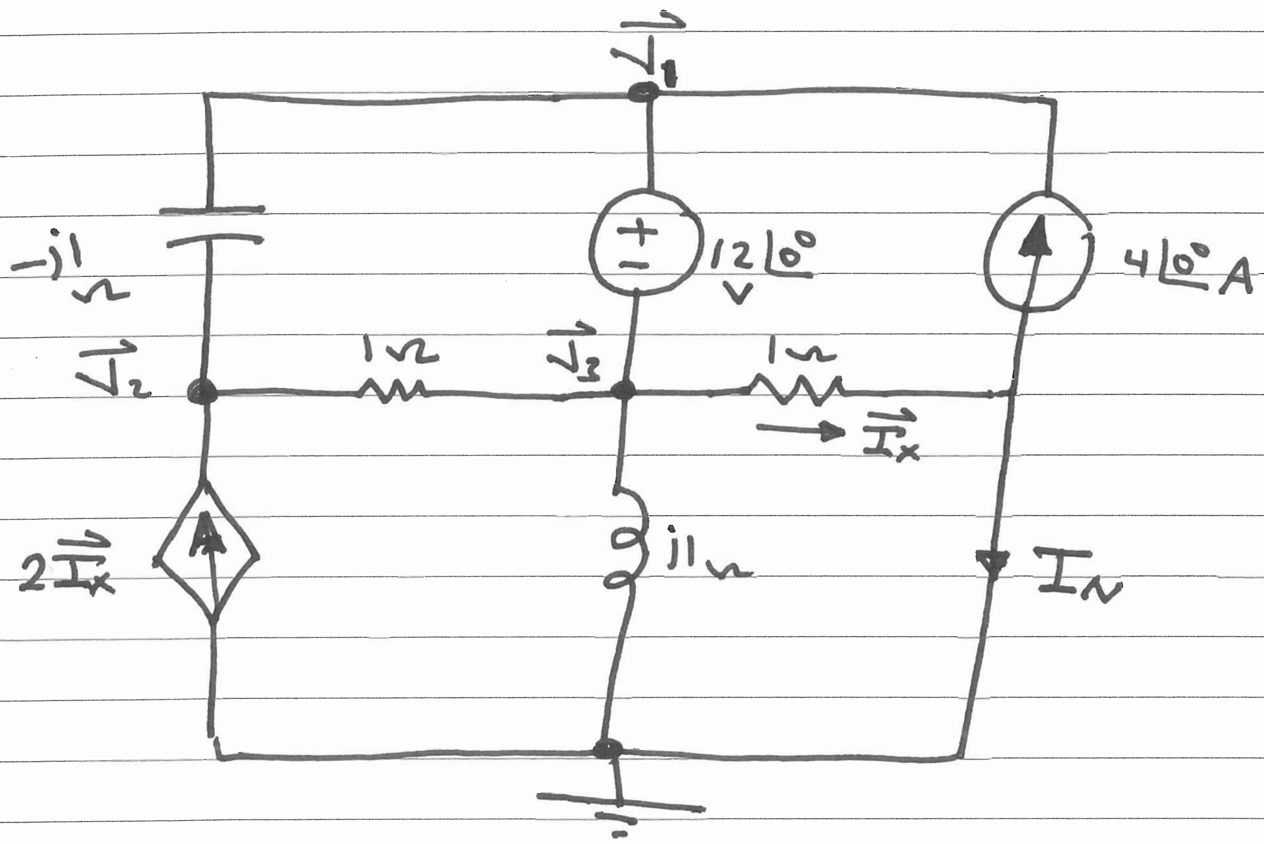
2) To find Z_{TH}

a) $Z_{TH} = \dfrac{\vec{V}_{TH}}{\vec{I}_N}$

b) $Z_{TH} = \dfrac{\vec{V}_T}{\vec{I}_T}\Bigg|$ all independent sources are set to zero

a) $Z_{TH} = \dfrac{\vec{V}_{TH}}{\vec{I}_N}$

To find $\vec{I}_N$



$$\vec{I}_N = \vec{I}_x - 4\angle 0^\circ$$

$$\vec{I}_x = \frac{\vec{V}_3}{1\Omega} = \vec{V}_3$$

Nodal Analysis

$$\vec{V}_2 - \vec{V}_3 = 12\angle 0^\circ \quad \text{Constrain equation}$$

KCL at node 2 :

$$2\vec{I}_x = \left(1 + \frac{1}{-j1}\right)\vec{V}_2 + jV_1 - 1V_3$$

KCL for the Supernode (1,3)

$$4\angle 0^\circ = \left(\frac{1}{-j1}\right)V_1 + \left(1+1+\frac{1}{j1}\right)V_3 - \left(1+\frac{1}{-j1}\right)V_2$$

Solving for $\vec{V_3}$

$$\vec{V_3} = \frac{4j}{1-j1}$$

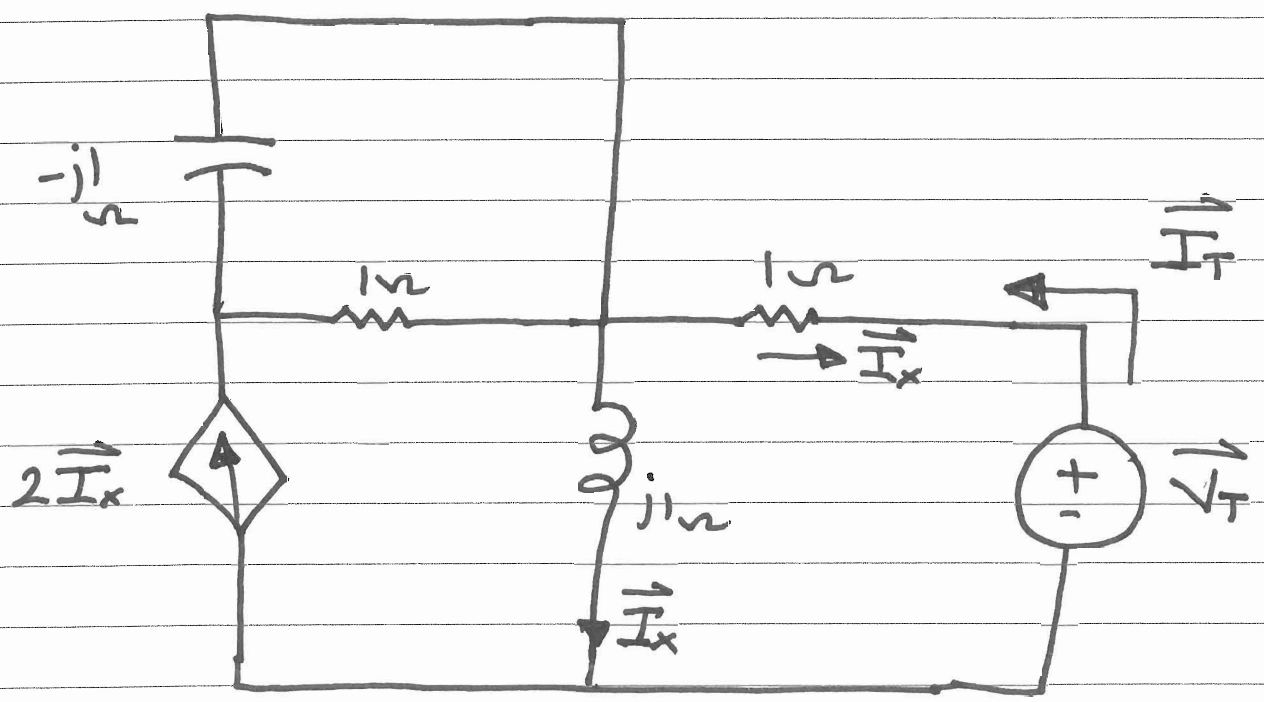
$$\therefore \; I_N = -\left(\frac{8+j4}{1+j1}\right)$$

$$\therefore \; Z_{TH} = \frac{\vec{V}_{TH}}{\vec{I}_N}$$

$$Z_{TH} = (1-j1)\,\Omega$$

$$\therefore \; \vec{V_o} = \frac{-4+j8}{1+1-j} = 4\angle 143.13^\circ \text{ V}$$

b) $Z_{TH} = \dfrac{\vec{V_T}}{\vec{I_T}}$ | independent source are zero



$$\vec{V_T} = -1(\vec{I_x}) + j1(\vec{I_x})$$

$$\vec{V_T} = (-1+j1)\vec{I_x}$$

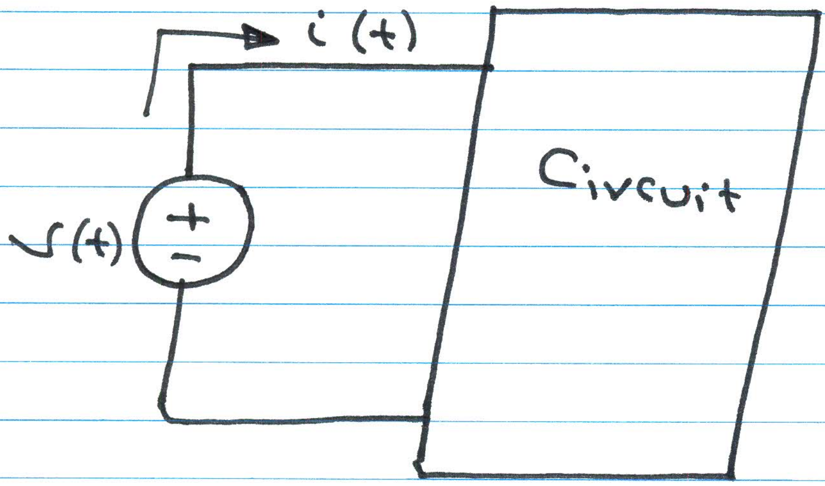
$$\vec{I_x} = -\vec{I_T}$$

$$\vec{V_T} = (-1+j1)\vec{I_T}$$

$$\therefore Z_{TH} = \frac{\vec{V_T}}{\vec{I_T}} = (1-j1)\ \Omega$$

Sinusoidal Steady State Power Calculation

Instantaneous Power : $p(t)$



$$v(t) = V_m \cos(\omega t + \theta_v)$$

$$i(t) = I_m \cos(\omega t + \phi_i)$$

$$p(t) = v(t)\, i(t)$$

$$p(t) = V_m I_m \cos(\omega t + \theta_v) \cos(\omega t + \phi_i)$$

$$\cos\alpha \cos\beta = \frac{1}{2}\left[\cos(\alpha - \beta) + \cos(\alpha + \beta)\right]$$

$$\therefore\ p(t) = \frac{V_m I_m}{2}\left[\cos(\theta_v - \phi_i) + \cos(2\omega t + \theta_v + \phi_i)\right]$$

Constant

Twice the excitation frequency

Example

$$v(t) = 4 \cos(\omega t + 60°) \quad V$$

$$Z(j\omega) = 2\angle 30° \quad \Omega$$

Find $p(t)$.

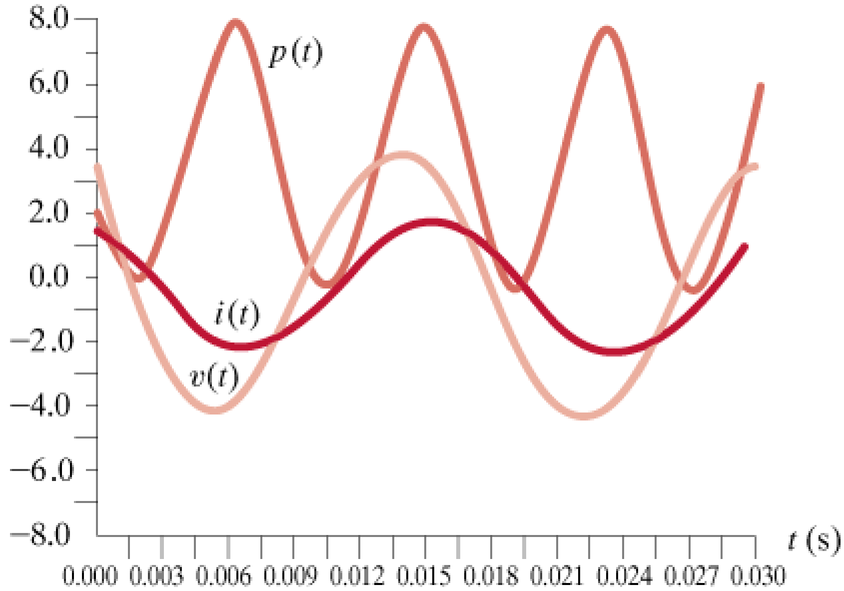
$$\vec{I} = \frac{\vec{V}}{Z} = \frac{4\angle 60°}{2\angle 30°} = 2\angle 30° \quad A$$

$$\therefore i(t) = 2 \cos(\omega t + 30°) \quad A$$

$$p(t) = v(t)\, i(t)$$

$$p(t) = 4 \cos 30° + 4 \cos(2\omega t + 90°)$$

$$p(t) = 3.46 + 4 \cos(2\omega t + 90°)$$



8.0
6.0
4.0
2.0
0.0
−2.0
−4.0
−6.0
−8.0
p(t)
i(t)
v(t)
t (s)
0.000 0.003 0.006 0.009 0.012 0.015 0.018 0.021 0.024 0.027 0.030

Average Power : Real Power

$$P_{av} = \frac{1}{T}\int_0^T p(t)\,dt$$

$$P_{av} = \frac{1}{2} V_m I_m \cos(\theta_v - \phi_i)$$

$$\theta_v - \phi_i = \theta_z$$

$$\therefore P_{av} = \frac{1}{2} V_m I_m \cos\theta_z$$

1) For Resistor

$$\theta_v - \phi_i = 0 \longrightarrow \theta_z = 0$$

$$\therefore P_{av} = \frac{1}{2} V_m I_m = \frac{V_m^2}{2R} = \frac{I_m^2 R}{2}$$

2) For Inductor

$$\theta_v - \phi_i = 90^\circ$$

$$P_{av} = 0$$

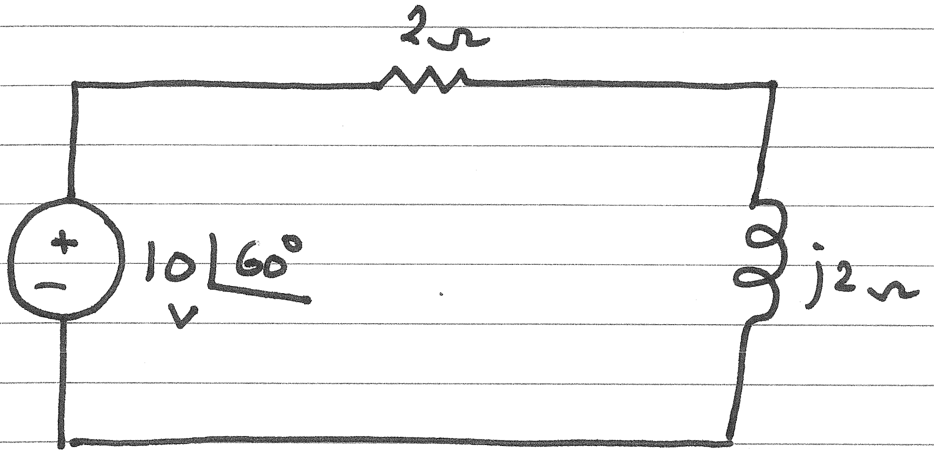
3) For Capacitor

$$\theta_v - \phi_i = -90^\circ$$

$\therefore \quad P_{av} = 0$

$\therefore$ Reactive impedances absorb no average power

Example



Find the average power absorbed by each elemen.

$$\vec{I} = \frac{10\angle 60^\circ}{2+j2} = 3.53\angle 15^\circ \quad A$$

$$P_{av} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$$P_{av_{j2\Omega}} = 0$$

$$P_{av_{2\Omega}} = \frac{I_m^2 R}{2} = \frac{(3.53)^2 \cdot 2}{2} = 12.5\,W$$

To Calculate the average power supplied by the Source

$$P_{av_{V_s}} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$I_m = 3.53\,A$

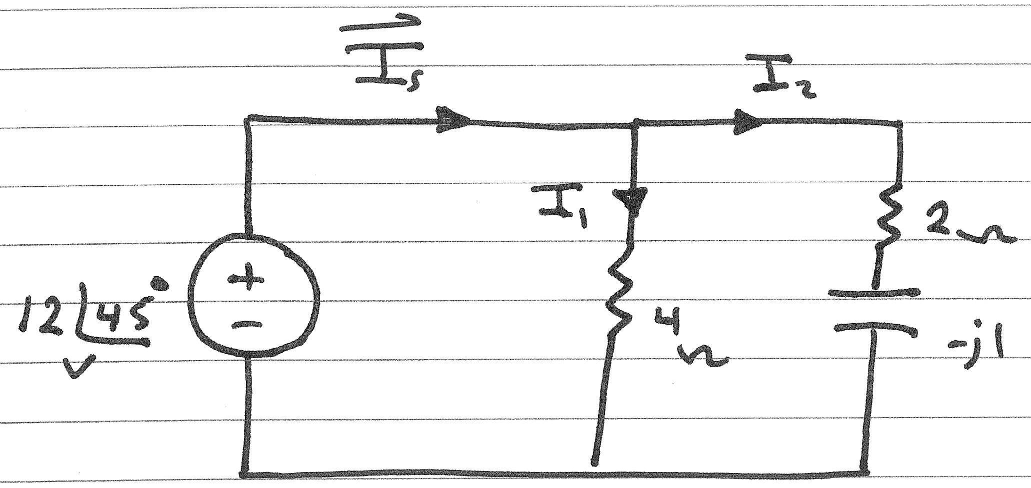
$V_m = 10\,V$

$\theta_v = 60^\circ$, $\phi_i = 15^\circ$

$$\therefore P_{av_{V_s}} = \frac{(10)(3.53)}{2} \cos(60 - 15^\circ)$$

$$= 12.5 \text{ Watt}$$

Example



Determine the average power absorbed by each resistor.

Determine the total average power absorbed and the average power supplied by the source.

$$\vec{I_1} = \frac{12\angle 45^\circ}{4\Omega} = 3\angle 45^\circ \text{ A}$$

$$\vec{I_2} = \frac{12\angle 45^\circ}{2 - j1} = 5.36\angle 71.57^\circ \text{ A}$$

$$\vec{I_s} = \vec{I_1} + \vec{I_2} = 8.15\angle 62.1^\circ \text{ A}$$

1) $P_{4\Omega} = \dfrac{I_{1m}^2 \cdot 4}{2} = 18\text{W}$

2) $P_{2\Omega} = \dfrac{I_{2m}^2 \cdot 2}{2} = 28.7 \text{ W}$

$\therefore$ Total Average power absorbed = 46.7 W

$$P_{V_s} = \frac{V_m I_m}{2} \cos\left(\theta_v - \phi_i\right)$$

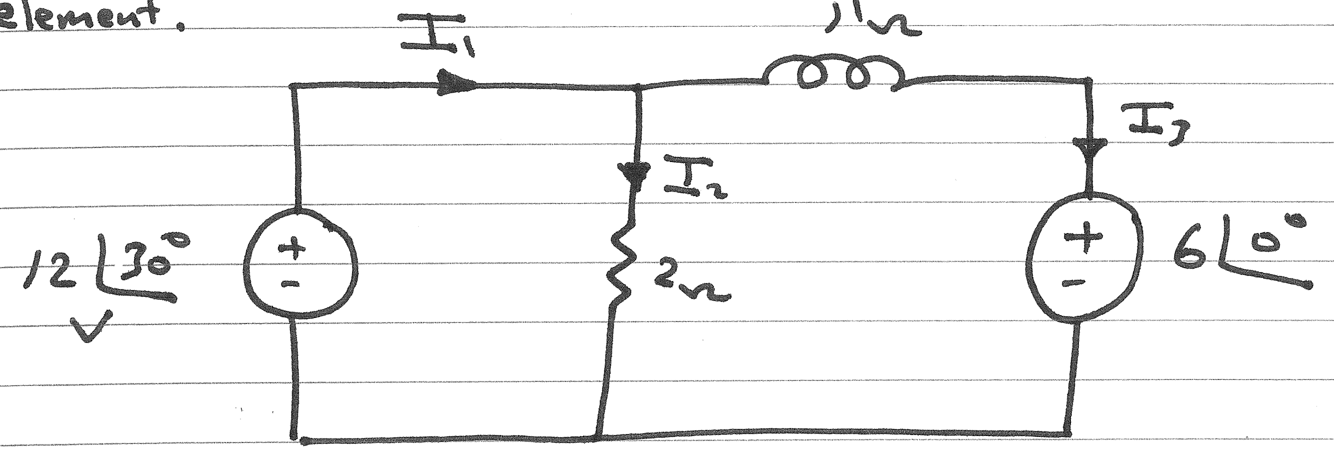
$$P_{V_s} = \frac{(12)(8.16)}{2} \cos\left(45 - 62.1\right)$$

$$P_{V_s} = 46.7\ W$$

$$\therefore \quad P_{V_s} = P_{4\Omega} + P_{2\Omega} + P_{-j1}$$

Example

Determine average power absorbed or supplied by each element.



$$\vec{I_2} = \frac{12\angle 30^\circ}{2} = 6\angle 30^\circ \ A$$

$$\vec{I_3} = \frac{12\angle 30^\circ - 6\angle 30^\circ}{j1} = 7.43\angle -36.19^\circ \ A$$

$$\vec{I_1} = \vec{I_2} + \vec{I_3} = 11.29\angle -7.07^\circ \ A$$

$$P_{2\Omega} = \frac{I_{2m}^2 \cdot 2}{2} = 36\ W$$

$$P_{12\angle 30^\circ} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$$P_{12\angle 30^\circ} = \frac{(12)(11.29)}{2} \cos\left(30 - (-7.07)\right)$$

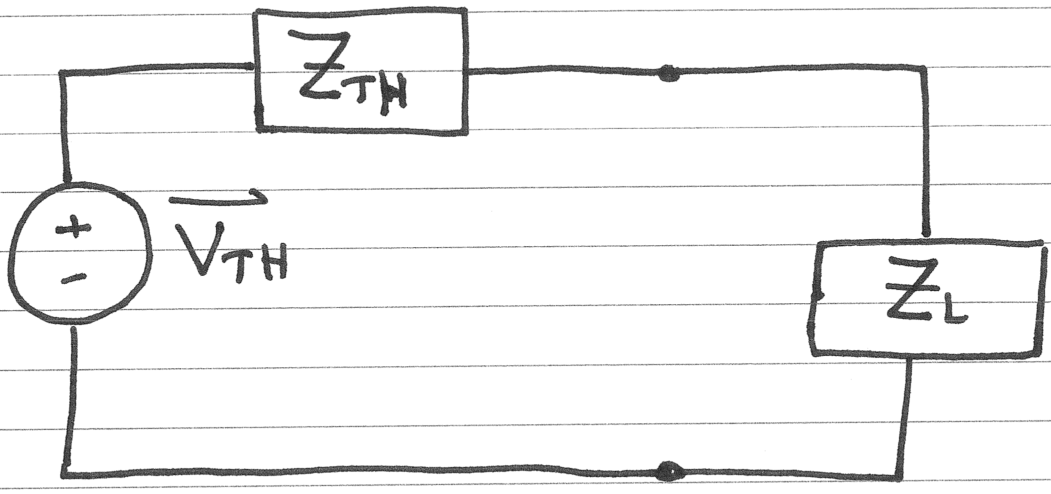
$$P_{12\angle 30^\circ} = 54\ W \quad \text{Supply}$$

$$P_{6\angle 0^\circ} = \frac{V_m I_m}{2} \cos\left(\theta_v - \phi_i\right)$$

$$P_{6\angle 0^\circ} = \frac{(6)(7.43)}{2} \cos\left(0 - (-36.19^\circ)\right)$$

$$P_{6\angle 0^\circ} = 18\ \text{W} \quad \text{absorbed}$$

Maximum Average power Transfer



$$Z_{TH} = R_{TH} + j\,X_{TH}$$

$$Z_L = R_L + j\,X_L$$

$$P_L = \frac{I_{Lm}^2 \cdot R_L}{2}$$

$$\vec{I} = \frac{\vec{V}_{TH}}{Z_{TH} + Z_L}$$

$$\vec{I} = \frac{\vec{V}_{TH}}{(R_{TH} + R_L) + j\,(X_{TH} + X_L)}$$

$$P_L = \frac{I_m^2 R_L}{2}$$

$$P_L = \frac{1}{2} \frac{V_{TH}^2 \cdot R_L}{(R_{TH}+R_L)^2 + (X_{TH}+X_L)^2}$$

$$\frac{\partial P_L}{\partial R_L} = 0 \quad ; \quad \frac{\partial P_L}{\partial X_L} = 0$$

$$\frac{\partial P_L}{\partial X_L} = \frac{-2 V_{TH}^2 R_L (X_L + X_{TH})}{2\left[(R_L+R_{TH})^2 + (X_L+X_{TH})^2\right]^2}$$

For $\frac{\partial P_L}{\partial X_L} = 0 \longrightarrow X_L = -X_{TH}$

$$\frac{\partial P_L}{\partial R_L} = \frac{V_{TH}^2\left[(R_L+R_{TH})^2 + (X_L+X_{TH})^2 - 2R_L(R_L+R_{TH})\right]}{2\left[(R_L+R_{TH})^2 + (X_L+X_{TH})^2\right]^2}$$

For $\frac{\partial P_L}{\partial R_L} = 0 \longrightarrow R_L = \sqrt{R_{TH}^2 + (X_L+X_{TH})^2}$

$$X_L = -X_{TH}$$

$$\therefore R_L = R_{TH}$$

$$\therefore Z_L = Z_{TH}^*$$

$$P_{L,max} = \frac{1}{8}\frac{V_{TH}^2}{R_L}$$

$\therefore$ For maximum average power Transfer

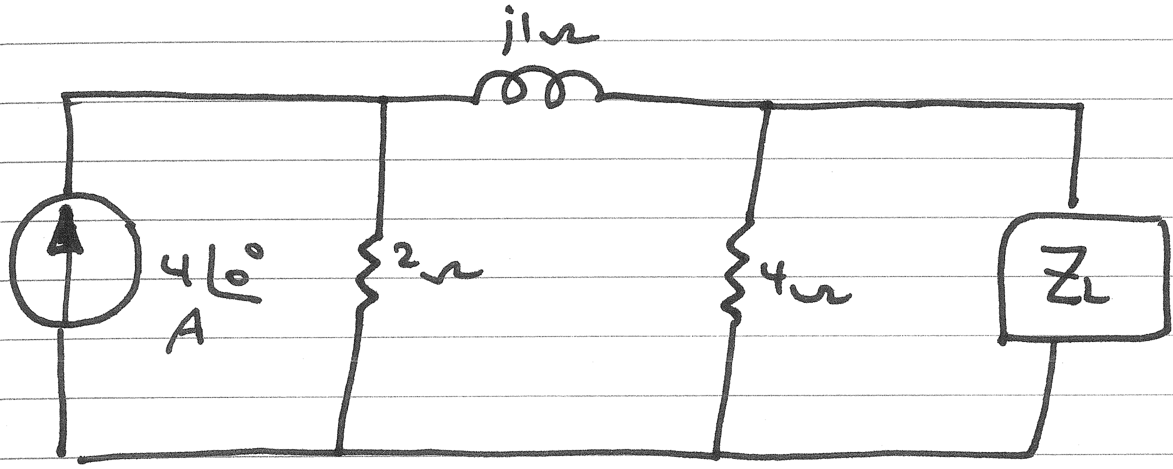
$$Z_L = Z_{TH}^{*}$$

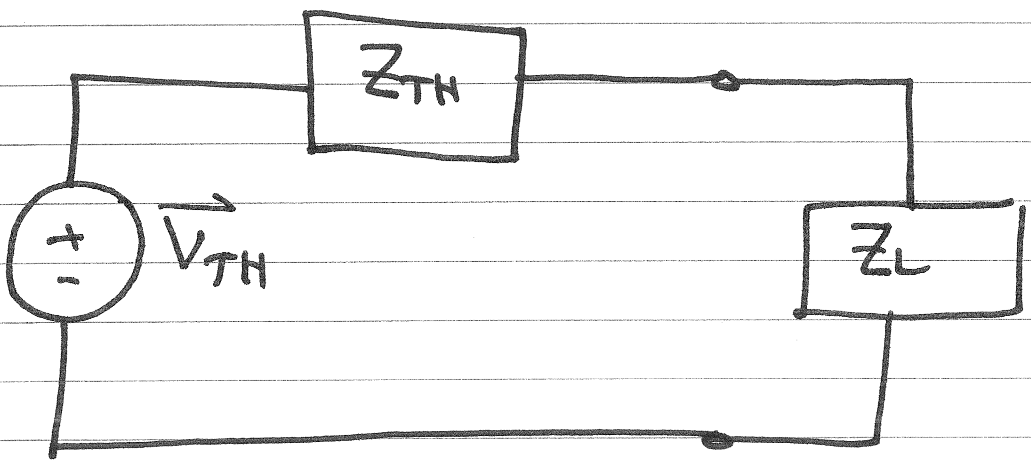
$$P_{L,max} = \frac{1}{8} \frac{V_{TH}^{2}}{R_{TH}}$$

Example

Find Z_L for maximum average power transfer.

Compute the maximum average power supplied to the load.





$$\vec{V}_{TH} = 4\angle 0^\circ \;\; \frac{2}{2 + j1 + 4} \cdot 4 = 5.28\angle -9.46^\circ \text{ V}$$

$$Z_{TH} = 4\,\Omega \parallel (2 + j1)\,\Omega$$

$$Z_{TH} = (1.4 + j\,0.43)\,\Omega$$

$$\therefore Z_L = (1.4 - j\,0.43)\,\Omega$$

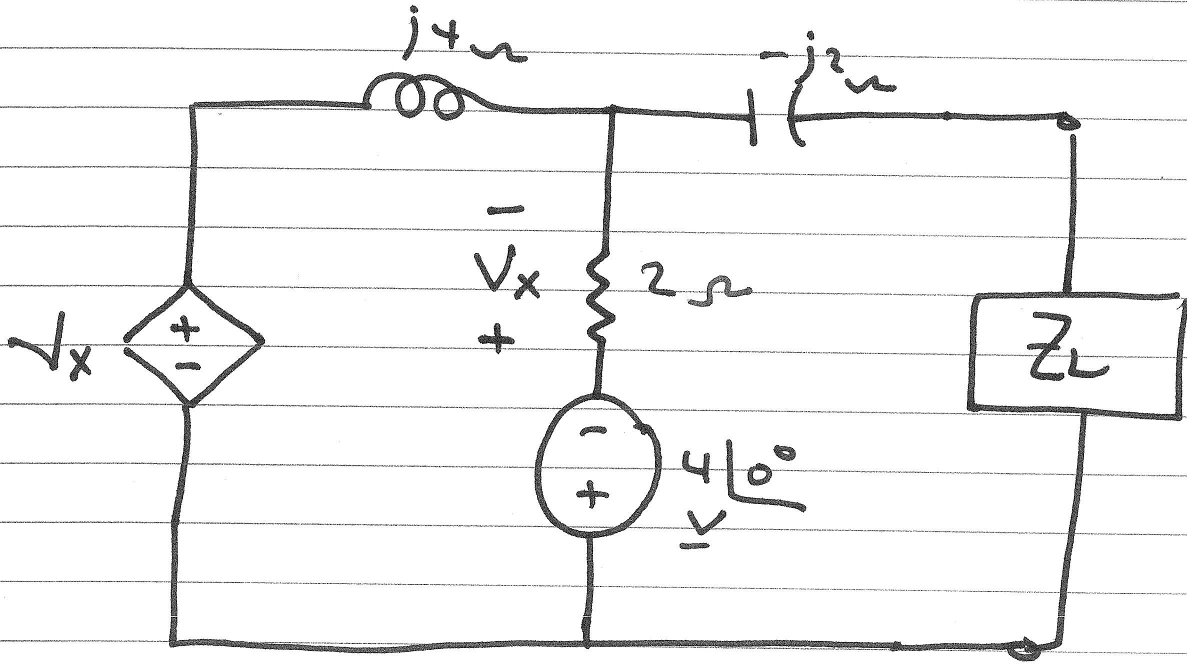
$$P_{,max} = \frac{1}{8}\,\frac{V_{TH}^2}{R_{TH}}$$

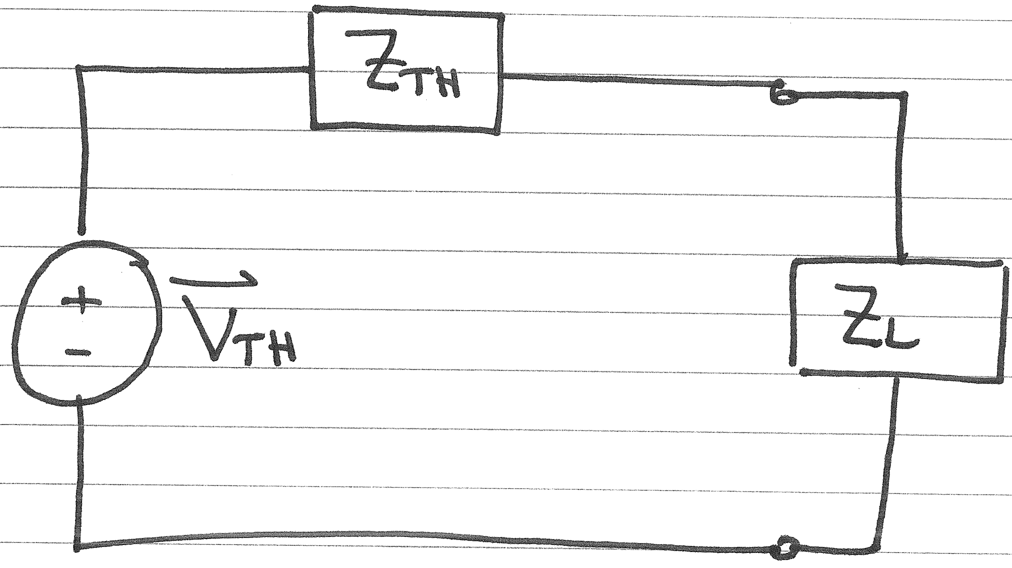
$$P_{,max} = 2.489\ W$$

Example

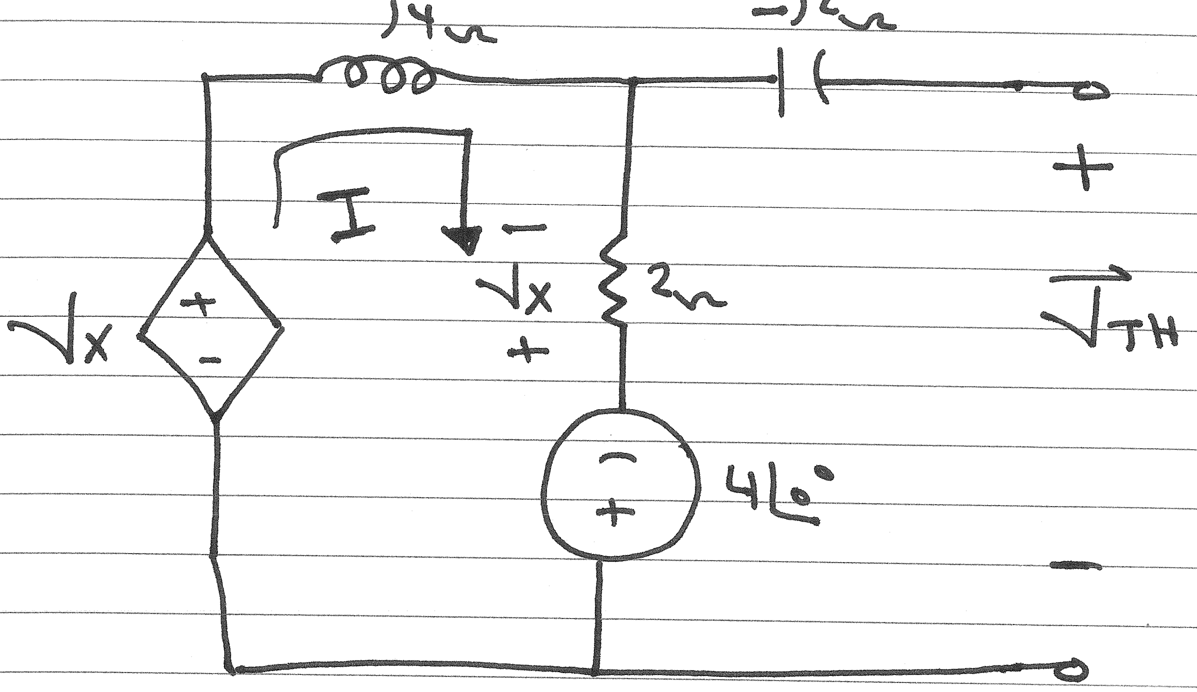
Find Z_L for maximum average power transfer

Compute the maximum average power supplied to Z_L





$$Z_L = Z_{TH}^{*}$$



$$\vec{V}_{TH} = 2\vec{I} - 4\angle 0^\circ$$

$$\vec{I} = \frac{V_x + 4\angle 0^\circ}{2 + j4}$$

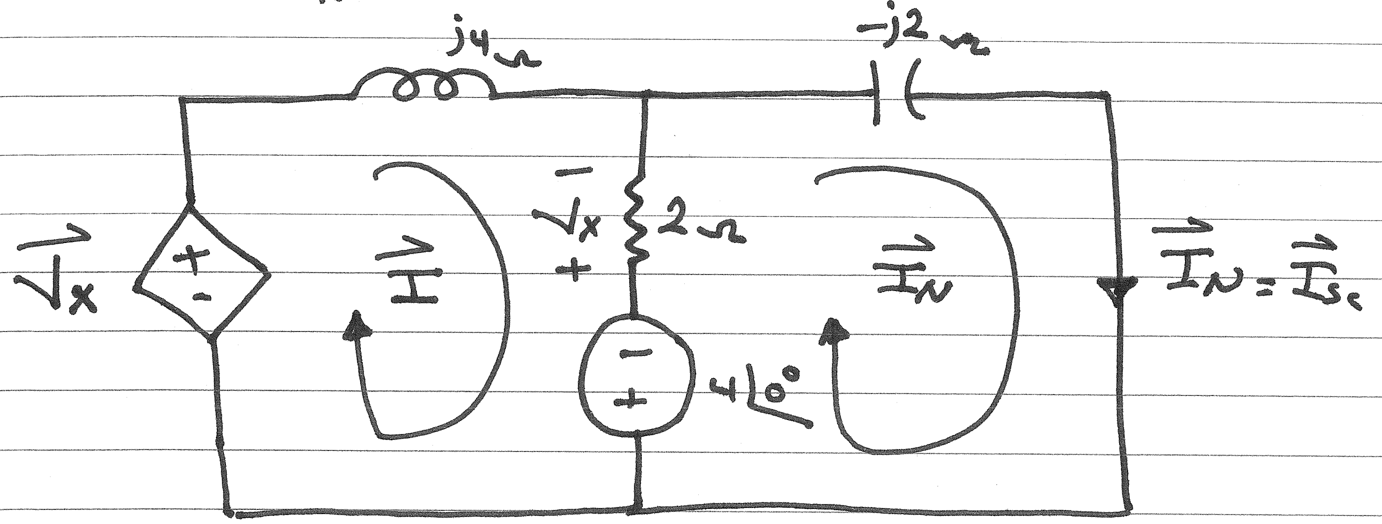
$$\vec{V}_x = -2\vec{I}$$

$$\vec{I} = 0.707\angle -45^\circ \ A$$

$$\therefore \vec{V}_{TH} = (-3 - j1)\,V$$

$$\vec{V}_{TH} = 3.16\angle 198.43^\circ \ V$$

$$Z_{TH} = \frac{\vec{V}_{TH}}{\vec{I}_N}$$



KVL for mesh 1 :

$$\vec{V}_x + 4\angle 0^\circ = (2+j4)\vec{I} - 2\vec{I}_N$$

$$\vec{V}_x = 2(\vec{I}_N - \vec{I})$$

KVL for mesh 2 :

$$-4\angle 0^\circ = -2\vec{I} + (2-j2)\vec{I}_N$$

Solving for $\vec{I}_N$

$$\vec{I}_N = (-1-j2)\,A$$

$$\vec{I}_N = 2.24\angle 243.43^\circ \;\; A$$

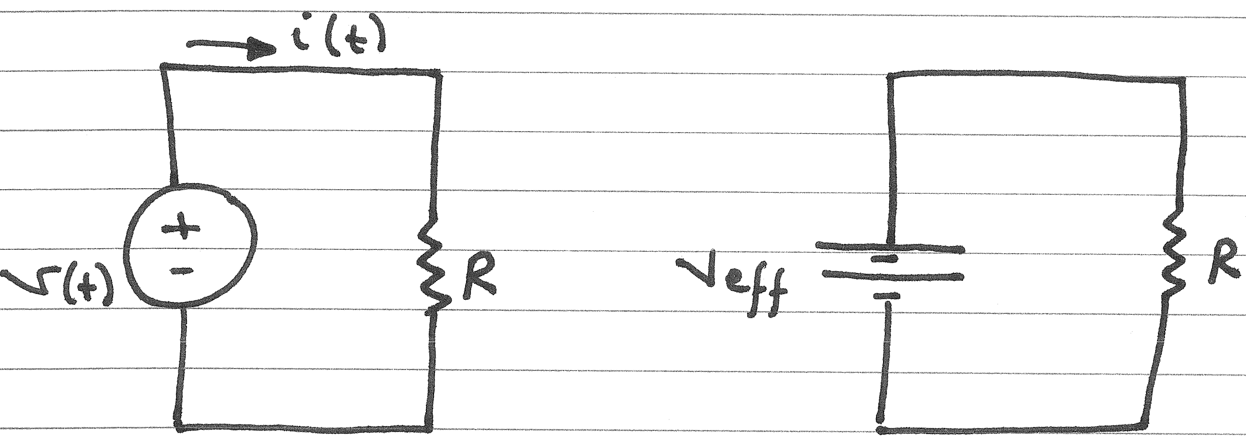
$$\therefore Z_{TH} = \frac{\vec{V}_{TH}}{\vec{I}_N} = 1.41\angle -45^\circ \;\Omega = (1-j1)\,\Omega$$

$$\therefore Z_L = Z_{TH}^* = 1.41\angle +45^\circ \;\Omega = (1+j1)\,\Omega$$

$$\therefore P_{L,max} = \frac{V_{TH}^2}{8R_{TH}} = 1.25\text{ W}$$

Effective or RMS value

The effective value of a periodic voltage (current) is the dc voltage (current) that delivers the same average power to a resistor as the periodic voltage (current).



let $v(t) = V_m \cos(\omega t + \theta_v)$

$$\therefore \quad P_1 = \frac{V_m^2}{2R}$$

$$P_2 = \frac{V_{eff}^2}{R}$$

$$P_1 = P_2$$

$$\therefore \quad \frac{V_m^2}{2R} = \frac{V_{eff}^2}{R}$$

$$\therefore V_{eff} = \frac{V_m}{\sqrt{2}}$$

RMS : Root Mean Square

let $v(t) = V_m \cos(\omega t + \theta_v)$

$$V_{RMS} = \sqrt{\frac{1}{T}\int_0^T V_m^2 \cos^2(\omega t + \theta_v)\,dt}$$

$$V_{RMS} = V_m \sqrt{\frac{1}{T}\int_0^T \cos^2(\omega t + \theta_v)\,dt}$$

$$V_{RMS} = V_m \sqrt{\frac{1}{T}\int_0^T \frac{1}{2}\left(1 + \cos 2(\omega t + \theta_v)\right)dt}$$

$$V_{RMS} = V_m \frac{1}{\sqrt{2}}$$

$$P_{av} = \frac{V_m I_m}{2} \cos(\theta_v - \phi_i)$$

$$P_{av} = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$

For a resistor

$$P_{av} = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$

$$V_{rms} = R I_{rms} \quad ; \quad \theta_v - \phi_i = 0$$

$$\therefore \quad P_{av} = \frac{V_{rms}^2}{R}$$

$$\therefore \quad P_{av} = I_{rms}^2 R$$

Apparent power and power factor

$$P_{av} = V_{rms}\, I_{rms} \cos(\theta_v - \phi_i)$$

$$P_{apparent} = V_{rms}\, I_{rms}$$

$P_{apparent}$ measured in VA

PF ≡ Power factor

$$PF = \cos(\theta_v - \phi_i)$$

$$\therefore \quad P_{av} = P_a \cdot PF$$

1) For Resistor

$$\theta_v - \phi_i = 0$$

$\therefore PF = 1$

2) For inductor

$$\theta_v - \phi_i = +90^\circ$$

$\therefore PF = 0$

3) For Capacitor

$$\theta_v - \phi_i = -90^\circ$$

$\therefore PF = 0$

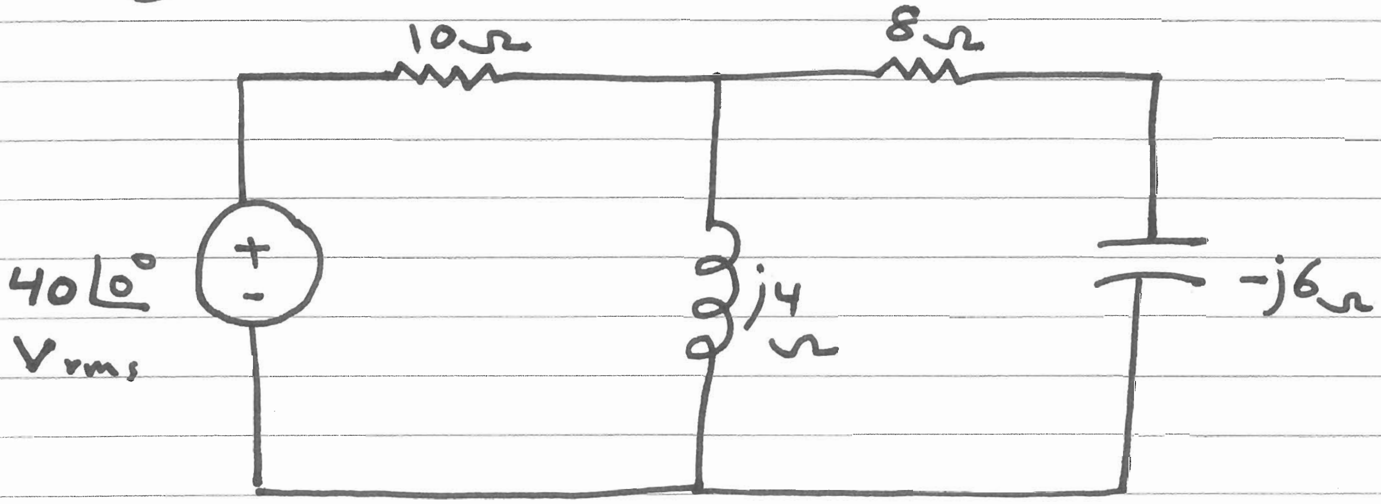
4) For Inductive Load

$90^\circ > \theta_v - \phi_i > 0$

$1 > PF > 0$ Lagging Power factor

Example

Calculate the power factor seen by the source and the average power supplied by the source.



$$Z = 10 + j4 \parallel (8 - j6)$$

$$Z = 12.69 \angle 20.62^\circ \quad \Omega$$

$$\vec{I}_s = \frac{40\angle 0^\circ}{Z} = 3.152 \angle -20.62^\circ \ A_{rms}$$

$$\Theta_v = 0^\circ \quad , \quad \Phi_i = -20.62^\circ$$

$$PF = \cos(\Theta_v - \Phi_i)$$

$$PF = \cos(20.62^\circ)$$

$$PF = 0.936 \text{ Lagging}$$

The average Power Supply by the Source is equal to the average power absorbed by the Circuit

---

$$P_{av} = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$

$V_{rms} = 40\ V_{rms}$

$I_{rms} = 3.152\ A_{rms}$

$\theta_v = 0^\circ$

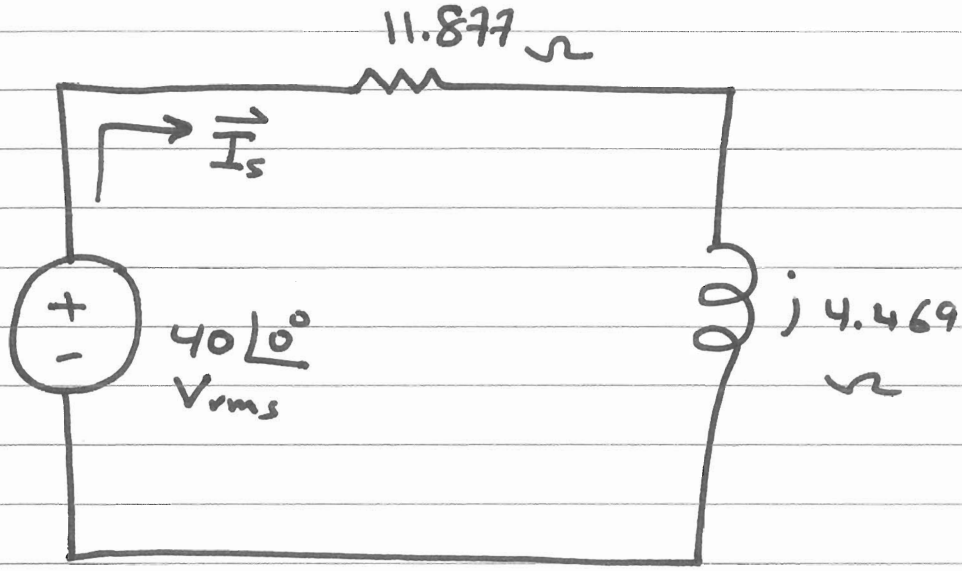
$\phi_i = -20.62^\circ$

$$\therefore P_{av} = (40)(3.152) \cos\left(0 - (-20.62^\circ)\right)$$

$$\therefore P_{av} = 118 \text{ Watt}$$

$$Z = 12.69\angle 20.62^\circ \ \Omega$$

$$Z = 11.877 + j\,4.469 \ \Omega$$



$$\therefore \quad P_{av} = I_{rms}^2 R$$

$$P_{av} = (3.152)^2 (11.877)$$

$$P_{av} = 118 \text{ W}$$

also $$P_{av} = P_{av_{10\Omega}} + P_{av_{8\Omega}} + P_{av_{-j6}} + P_{av_{j4}}$$

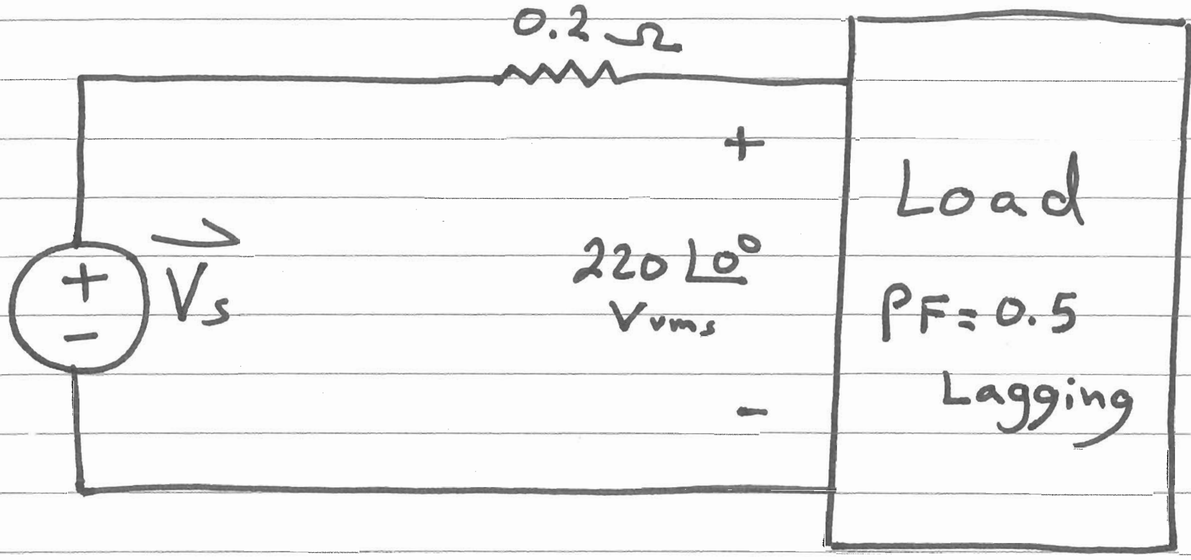
$$P_{av} = P_{av_{10\Omega}} + P_{av_{8\Omega}}$$

Example

An industrial Load Consumes 11 KW at 0.5 PF Lagging from a 220 V rms Line. The transmission Line resistance from the power Company to the plant is $0.2\ \Omega$.

1) Determine the average power that must be supplied by the power Company

2) Repeat ① if the power factor is changed to unity.



$$P_{av_L} = V_{rms} \cdot I_{rms} \cdot PF$$

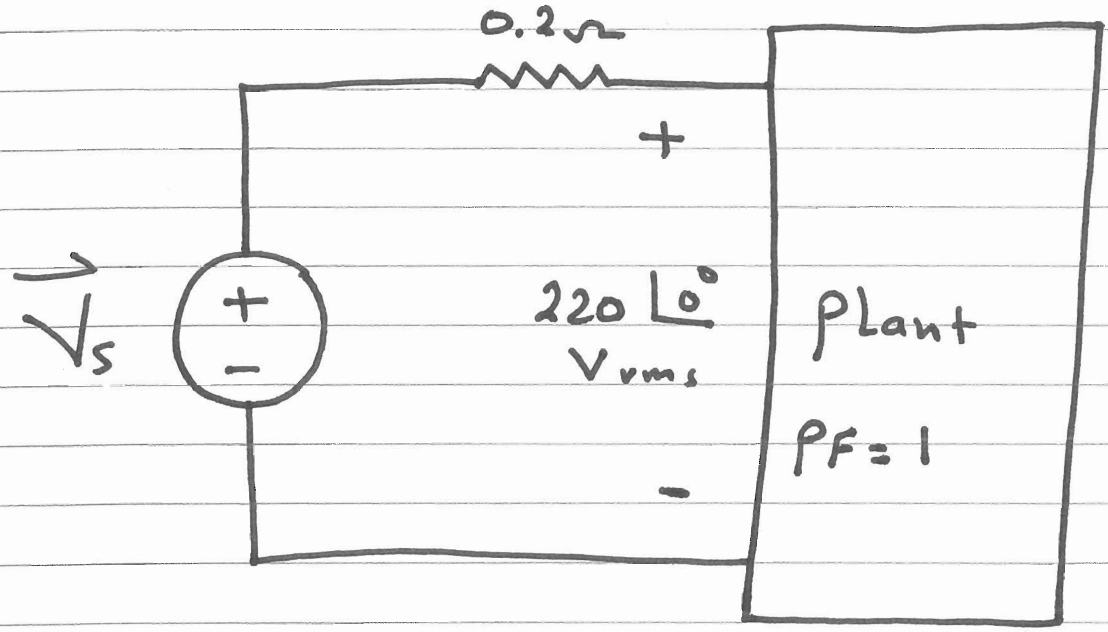
$$\therefore \; I_{rms} = \frac{P_{av_L}}{V_{rms} \cdot PF}$$

$$I_{rms} = \frac{11\,KW}{(220)(0.5)} = 100 \; A_{rms}$$

$$P_{Loss} = (I_{rms}^2) \cdot (0.2) = 2\,KW$$

$$\therefore \; P_{av_{sup}} = P_{av_L} + P_{av_{Loss}}$$

$$P_{av_{sup}} = 13\,KW$$



$$P_{av_{Load}} = V_{rms} \cdot I_{rms} \cdot PF$$

$$\therefore \quad I_{rms} = \frac{P_{av}}{V_{rms} \cdot PF} = 50\, A_{rms}$$

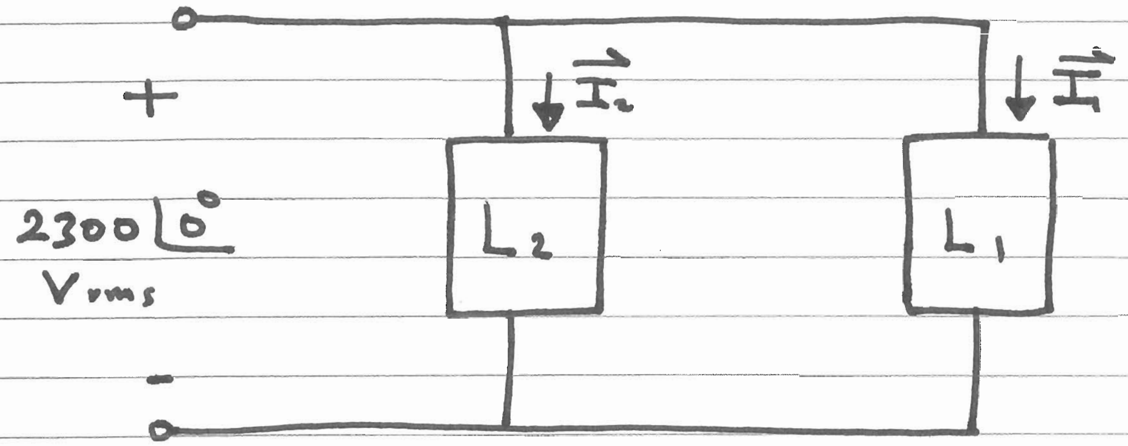
$$P_{Loss} = I_{rms}^2 \cdot R = (50)^2 \cdot (0.2) = 0.5\, KW$$

$$\therefore \quad P_{av_{sup}} = 0.5\, KW + 11\, KW$$

$$P_{av_{sup}} = 11.5\, KW$$

Example

Find the power factor of the two Loads



Load 1 : 10 KW , 0.9 Lagging PF

Load 2 : 5 KW , 0.95 Leading PF

$$\vec{I_1} = \frac{10{,}000}{(2300)(0.9)} \angle -\cos^{-1} 0.9$$

$$\vec{I_1} = 4.83 \angle -25.84^\circ \quad A_{rms}$$

$$\vec{I_2} = \frac{5000}{(2300)(0.95)} \angle +\cos^{-1} 0.95$$

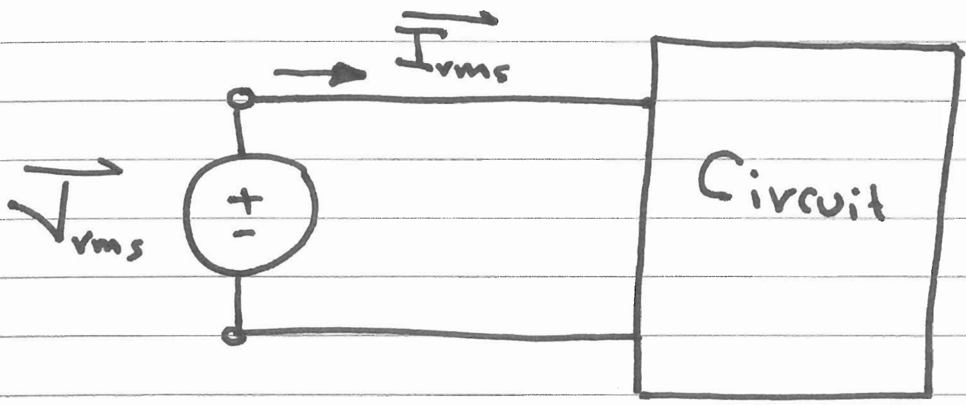
$$\vec{I_2} = 2.288 \angle 18.195^\circ \quad A_{rms}$$

$$\vec{I_s} = \vec{I_1} + \vec{I_2} = 6.78 \angle -12^\circ \quad A_{rms}$$

$$PF = \cos(\theta_v - \theta_i) = \cos(0 - (-12^\circ))$$

$$PF = 0.978 \text{ Lagging}$$

Complex Power



$$\vec{V}_{rms} = V_{rms} \angle \theta_v$$

$$\vec{I}_{rms} = I_{rms} \angle \phi_i$$

$\vec{S} \equiv$ Complex power

$$\vec{S} = \vec{V}_{rms} \cdot \vec{I}_{rms}^{\,*}$$

$$\vec{S} = V_{rms} I_{rms} \angle \theta_v - \phi_i$$

$$\vec{S} = V_{rms} I_{rms} \cos(\theta_v - \phi_i) + j V_{rms} I_{rms} \sin(\theta_v - \phi_i)$$

$$\vec{S} = P_{av} + j Q$$

$$\vec{S} = P_{av} + jQ$$

$P_{av} \equiv$ Average power in Watt

$Q \equiv$ Reactive Power in VAR

$\therefore \quad P_{av} = \text{Real}\left[\vec{S}\right]$

$Q = \text{Imajinary}\left[\vec{S}\right]$

$Q = V_{rms} I_{rms} \sin(\theta_v - \phi_i)$

1) For pure resistance

$\theta_v - \phi_i = 0$

$\therefore \quad Q_R = 0$

2) For pure inductance

$\theta_v - \phi_i = +90^\circ$

$\therefore \quad Q_L = V_{rms} I_{rms}$

$V_{rms} = WL \, I_{rms}$

$\therefore \quad Q_L = WL \, I_{rms}^2 = \dfrac{V_{rms}^2}{WL}$

3) For pure Capacitance

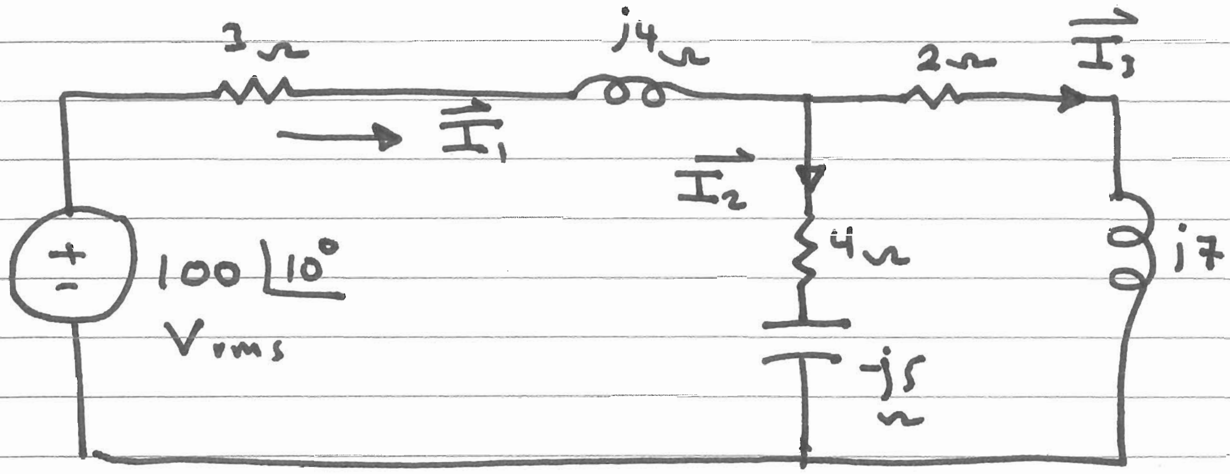
$$\theta_v - \phi_i = -90^\circ$$

$$Q_c = -V_{rms} I_{rms}$$

$$I_{rms} = \omega C V_{rms}$$

$$\therefore Q_c = -\frac{I_{rms}^2}{\omega C} = -\omega C V_{rms}^2$$

What are the VARS Consumed by the Circuit



$$Q = V_{rms}\, I_{rms} \sin(\theta_v - \phi_i)$$

$$\vec{I}_1 = \frac{\vec{V}_s}{Z}$$

$$Z = (2+j7)\,\|\,(4-j5) + 3 + j4$$

$$Z = 10.35 + j4.55 = 11.3\angle 23.7^\circ \ \Omega$$

$$\therefore \vec{I}_1 = \frac{100\angle 10^\circ}{11.3\angle 23.7^\circ} = 8.84\angle -13.7^\circ \ A_{rms}$$

$$\therefore Q = (100)(8.84)\sin\left(10 - (-13.7^\circ)\right)$$

$$\therefore Q = 355 \text{ VARS}$$

$$I_2 = 10.2 \ A_{rms}$$

$$I_3 = 8.95 \ A_{rms}$$

$$P_{av} = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$
$$Q = V_{rms} I_{rms} \sin(\theta_v - \phi_i)$$

$$\frac{Q}{P_{av}} = \tan(\theta_v - \phi_i)$$

$$\therefore \; Q = P_{av} \tan(\theta_v - \phi_i)$$

$$Q = P_{av} \tan\left[\cos^{-1}(PF)\right]$$

$$\vec{S} = P_{av} + jQ$$

$$\vec{S} = \sqrt{P_{av}^2 + Q^2} \; \angle \tan^{-1}\frac{Q}{P_{av}}$$

$$\vec{S} = V_{rms} I_{rms} \; \angle \theta_v - \phi_i$$

$$\therefore \; P_a = |S| = \sqrt{P_{av}^2 + Q^2}$$

$$\theta_v - \phi_i = \tan^{-1}\frac{Q}{P_{av}}$$

$$\theta_v - \phi_i = \tan^{-1} \frac{Q}{P_{av}}$$

To increase P.F , we need to decrease Q

∴ For inductive circuit , we add a Capacitor in parallel to increase the power factor

$$P_{av_T} = P_{av_1} + P_{av_2} + \cdots P_{av_n}$$

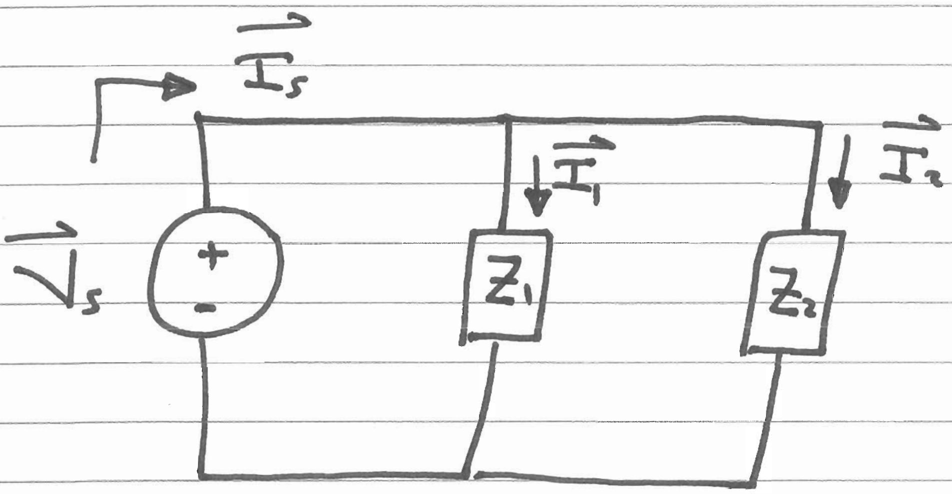
$$Q_T = Q_1 + Q_2 + \cdots Q_n$$

$$\vec{S}_T = P_{av_T} + j\, Q_T$$

$$\vec{S}_T = \vec{S}_1 + \vec{S}_2 + \cdots \vec{S}_n$$

Conservation of Ac Power

The Complex, real, and reactive powers of the Sources equal the respective Sum of the Complex, real, and reactive powers of the individual Loads.



$$\vec{S}_{source} = \vec{V}_s \cdot \vec{I}_s^{\,*}$$

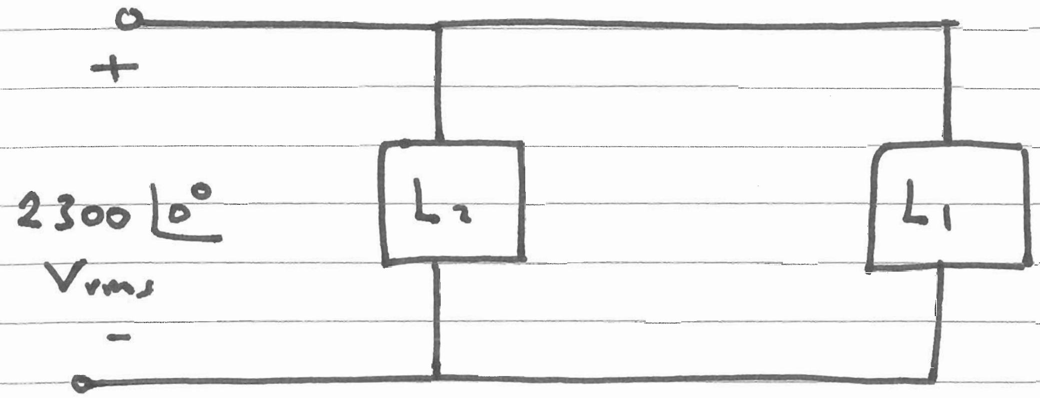
$$\vec{S}_{source} = \vec{V}_s \cdot \left(\vec{I}_1^{\,*} + \vec{I}_2^{\,*}\right)$$

$$\vec{S}_{source} = \vec{V}_s \cdot \vec{I}_1^{\,*} + \vec{V}_s \cdot \vec{I}_2^{\,*}$$

$$\vec{S}_{source} = \vec{S}_1 + \vec{S}_2$$

The Same results Can be obtained for a series Connection.

Find the power factor of the two Loads



Load 1 : 10 KW , 0.9 Lagging P.F

Load 2 : 5 KW , 0.95 Leading P.F

$$\vec{S_1} = P_{av_1} + j\,Q_1$$

$$Q_1 = P_{av_1} \tan\left[\cos^{-1}(P.F._1)\right] = 4843 \text{ VARS}$$

$$\therefore \vec{S_1} = 10000 + j\,4843$$

$$\vec{S_2} = P_{av_2} + j\,Q_2$$

$$Q_2 = -P_{av_2} \tan\left[\cos^{-1}(P.F._2)\right]$$

$$Q_2 = -1643 \text{ VARS}$$

$$\therefore \vec{S_2} = 5000 - j\,1643$$

$$\vec{S}_T = \vec{S}_1 + \vec{S}_2$$

$$\vec{S}_T = 15000 + j\,3200$$

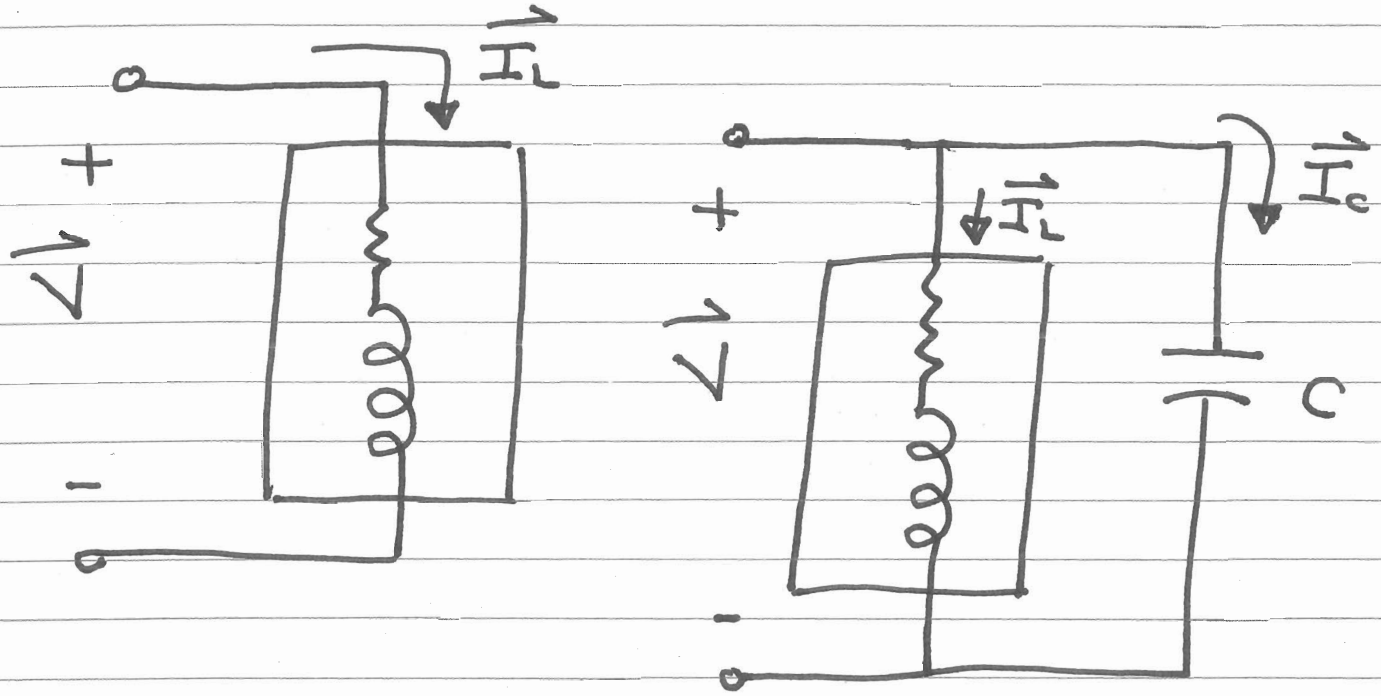
$$\vec{S}_T = 15337.5 \angle 12.02^\circ \quad VA$$

$$P.F = \cos(12.02)$$

$$P.F = 0.978 \text{ Lagging}$$

Power Factor Correction

Power factor Correction is the process of increasing the power factor without altering the voltage or current to the original load.



Power factor Correction is necessary for Economic Reason.

$$PF = \cos(\theta_v - \phi_i)$$

For R

$$P.F = 1$$

$$Q_R = 0$$

∴ To improve the power factor we must decrease the Reactive power

∴ For inductive Circuit, we add a Capacitor in parallel to the load.

$$Q_c = Q_{Final} - Q_{init}$$

$$C = -\frac{Q_c}{\omega V_{rms}^2}$$

Example

A certain industrial plant Consumes 1 MW at 0.7 Lagging power factor and a 2300 V rms.

What is the minimum Capacitor required to improve the power factor to 0.9 Lagging. $\omega = 377$ r/s

---

$$Q_{ini} = P_{av} \tan\left[\cos^{-1} PF_1\right]$$

$$Q_{ini} = 1M \tan\left[\cos^{-1} 0.7\right]$$

$$Q_{ini} = 1.02 \text{ MVARS}$$

$$Q_{Fin} = P_{av} \tan\left[\cos^{-1} PF_2\right]$$

$$Q_{Fin} = P_{av} \tan\left[\cos^{-1} 0.9\right]$$

$$Q_{Fin} = 0.484 \text{ MVARS}$$

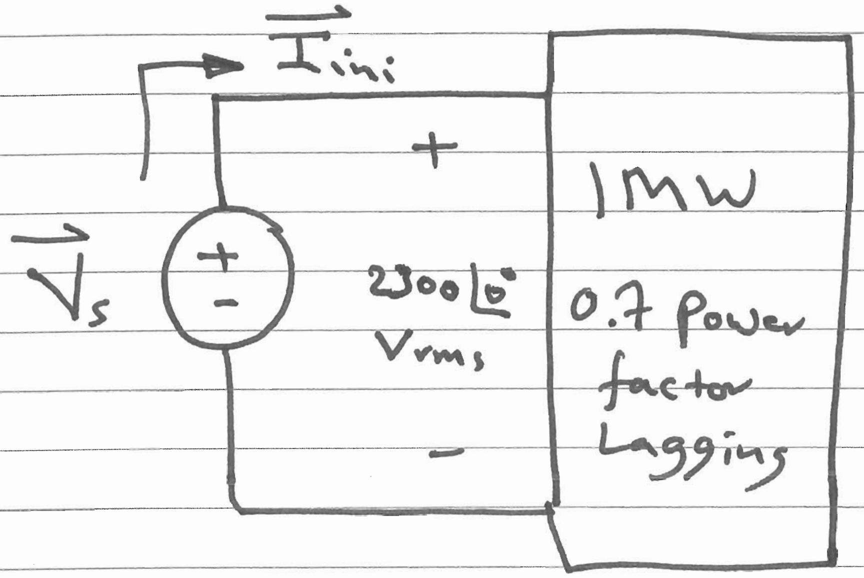
$$Q_c = Q_{Fin} - Q_{ini}$$

$$Q_c = -0.536 \text{ MVARS}$$

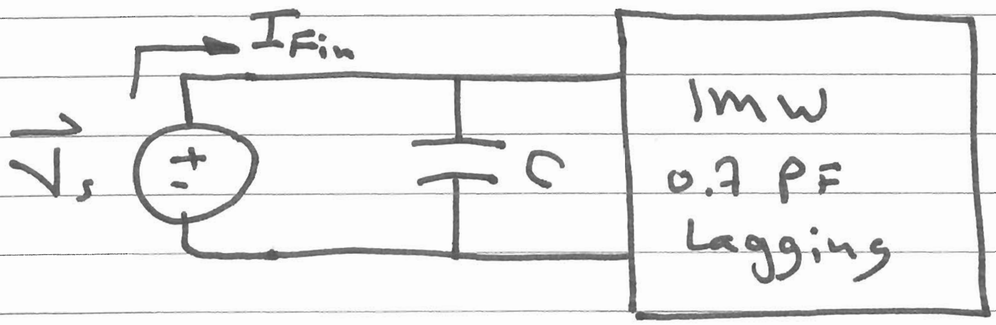
$$Q_c = -\frac{V_{rms}^2}{X_c}$$

$$Q_c = -\omega C V_{rms}^2$$

$$\therefore C = -\frac{Q_c}{\omega V_{rms}^2} = 269\ \mu F$$

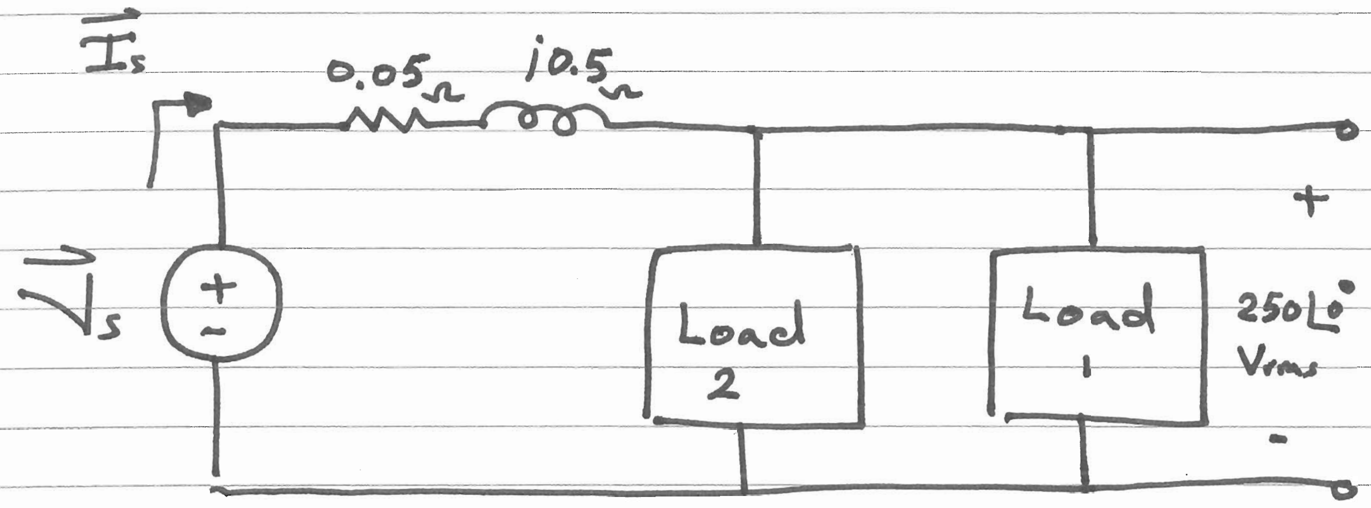


$$I_{ini} = \frac{P_{av}}{(V_{rms})(PF_1)} = 621\ A_{rms}$$



$$I_{Fin} = \frac{P_{av}}{(V_{rms})(PF_2)} = 483\ A_{rms}$$

Example



Load 1 : 8 KW ; 0.8 PF Leading

Load 2 : 20 KVA ; 0.6 PF Lagging

1) Determine the power factor of the two Loads in parallel

2) Determine the apparent power required to supply the loads; the magnitude of the current I_s ; the average power loss in the transmission Line

3) Compute the value of the capacitor that would correct the power factor to 1 if

placed in parallel with the two loads

$\omega = 377\ r/s$

4) Repeat step 2

Load 1 : 8 KW ; 0.8 PF_1 leading

Load 2 : 20 KVA ; 0.6 PF_2 Lagging

$$P_{av_1} = 8000 \text{ W}$$

$$\therefore Q_1 = -P_{av_1} \tan\left[\cos^{-1}(PF_1)\right] = -6000 \text{ VARS}$$

$$\therefore \vec{S_1} = P_{av_1} + j\,Q_1$$

$$\vec{S_1} = 8000 - j\,6000 \quad \text{VA}$$

$$P_{a_2} = 20000 \text{ VA} \; ; \; PF_2 = 0.6 \text{ Lagging}$$

$$\therefore P_{av_2} = P_a \cdot PF_2 = 12000 \text{ W}$$

$$Q_2 = P_{av_2} \tan\left[\cos^{-1}(PF_2)\right) = +16000 \text{ VARS}$$

$$\therefore \vec{S_2} = P_{av_2} + j\,Q_2$$

$$\vec{S_2} = 12000 + j\,16000 \quad \text{VA}$$

$$\vec{S_{T_L}} = \vec{S_1} + \vec{S_2}$$

$$\vec{S_{T_L}} = 20000 + j\,10000 \quad \text{VA}$$

$$\vec{S_{T_L}} = 22360 \underline{\lfloor 26.565^\circ} \quad \text{VA}$$

$$\therefore PF_{Loads} = \cos(26.565) = 0.8944 \text{ Lagging}$$

$$\vec{S}_{L_T} = \vec{V}_{rms}\,\vec{I}_s^{*}$$

$$\therefore \vec{I}_s^{*} = \frac{\vec{S}_{L_T}}{\vec{V}_{rms}} = \frac{22360}{250} \angle 26.565^\circ$$

$$\therefore \vec{I}_s = 89.44 \angle -26.565^\circ \quad A\ vms$$

Since $\vec{S}_{L_T} = 22360 \angle 26.565^\circ$

$$\therefore P_a = |S_{L_T}| = 22360 \quad VA$$

$$P_{av\ Loss} = |\vec{I}_s|^2 \cdot (0.05)$$

$$= 400\,W$$

3) Since $\vec{S}_{L_T} = 20000 + j\,10000$

$$\therefore Q_{ini} = 10000 \quad VARS$$

$$Q_{Fin} = 0$$

$$\therefore Q_c = Q_{Fin} - Q_{ini} = -10000\,VARS$$

$$\therefore C = \frac{-Q_c}{w V_{rms}^2} = 424.4\,\mu F$$

4) Since $Q_{Fin} = 0$

$\therefore \vec{S_F} = P_{av} = 20000$ VA

$\therefore P_a = P_{av} = 20000$ VA

$\therefore \vec{S_F} = 20000 \angle 0^\circ$ VA

$\vec{S_F} = \vec{V_{rms}} \cdot \vec{I_s^*}$

$\therefore \vec{I_s^*} = \dfrac{20000 \angle 0^\circ}{250 \angle 0^\circ} = 80 \angle 0^\circ$ Arms

$\therefore \vec{I_s} = 80 \angle 0^\circ$ A rms

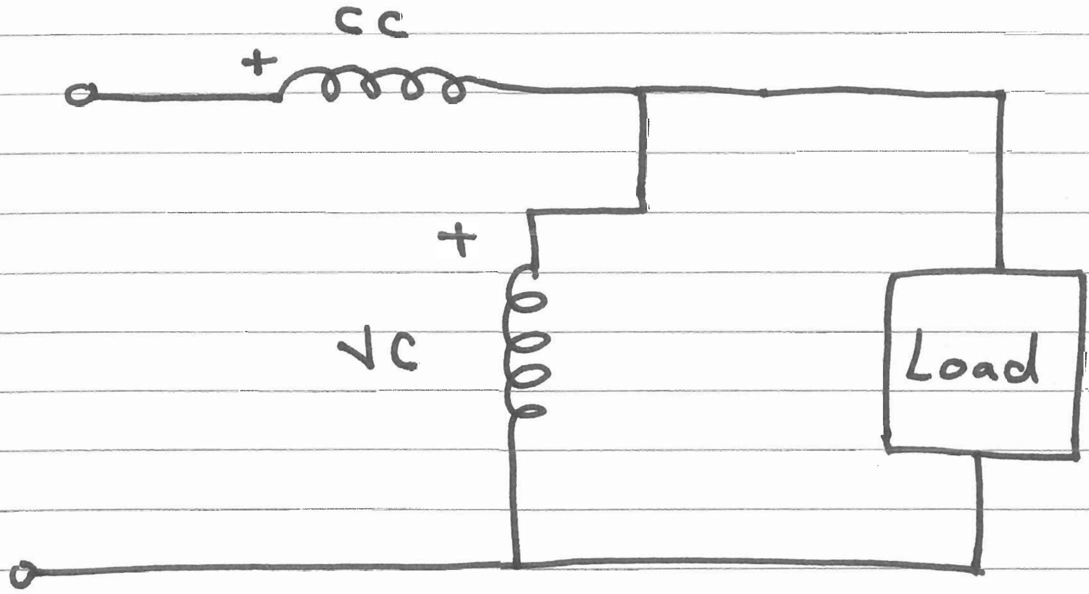
$P_{Loss} = |I_s|^2 \cdot (0.05)$

$P_{Loss} = 320$ W

Power Measurement

Wattmeter is the instrument for measuring the average power

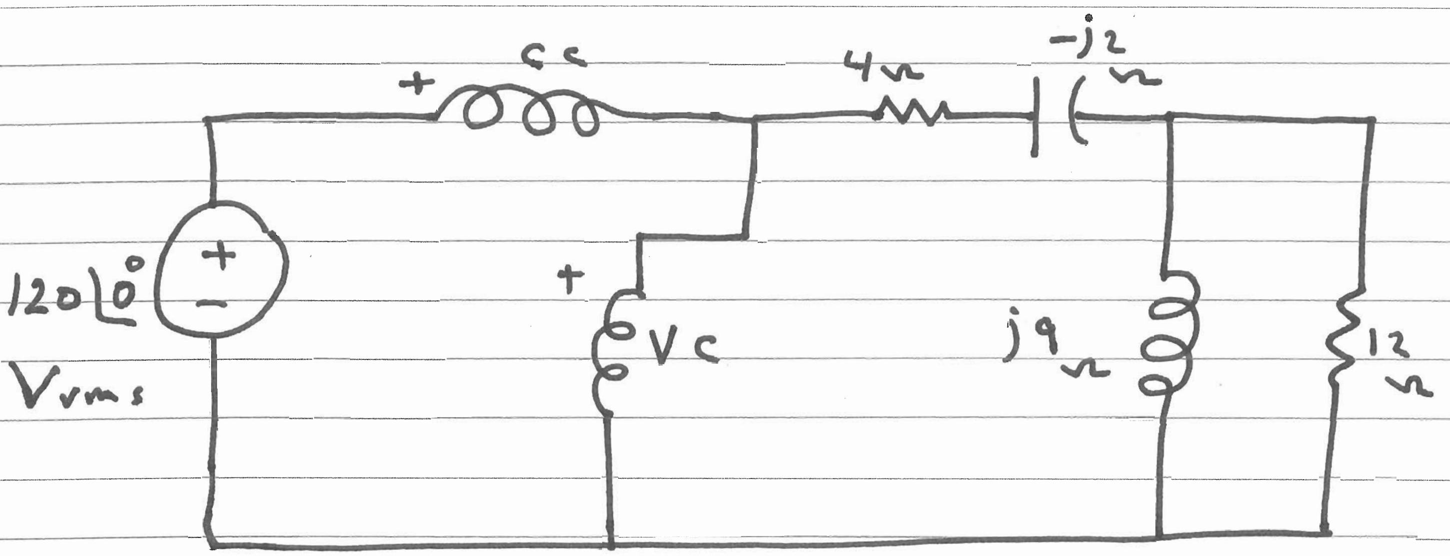
Two Coils are used, the high impedance Voltage Coil and the Low impedance Current Coil.



$$P = V_{rms} I_{rms} \cos(\theta_v - \phi_i)$$

Example

Find the Wattmeter reading



$$Z = 4 - j2 + (j9 \| 12)$$

$$Z = 9.13 \angle 24.32^\circ$$

$$\vec{I} = \frac{120 \angle 0^\circ}{9.13 \angle 24.32^\circ} = 13.14 \angle -24.32^\circ \text{ A}_{rms}$$

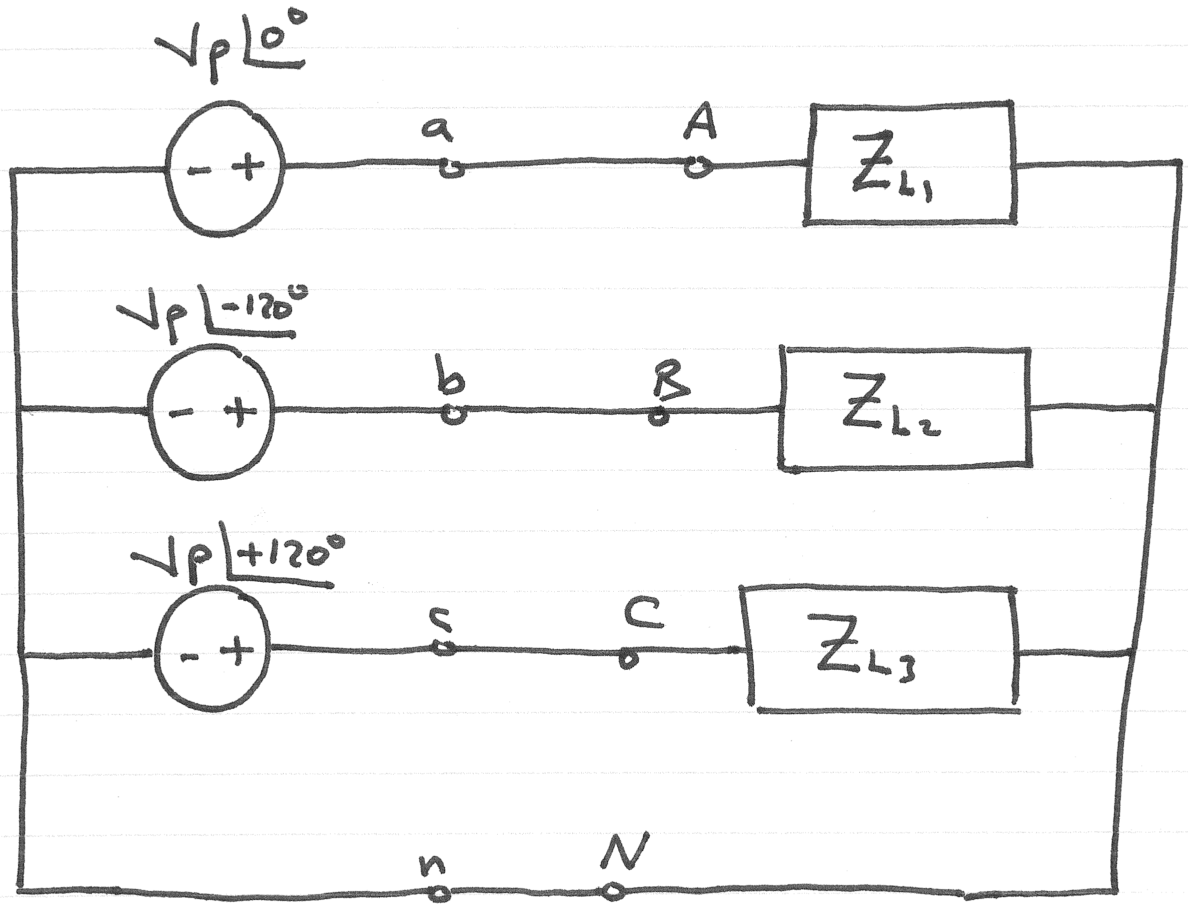
$$P = (120)(13.14) \cos(0 + 24.32^\circ)$$

$$P = 1436.9 \text{ W}$$

Balanced Three-phase Circuits

What is a Three-phase Circuit ?

It is a system produced by a generator Consisting of three sources having the same amplitude and frequency but out of phase with each other by 120°.



Advantages :

1. Almost all the electric power is generated and distributed in three-phase.

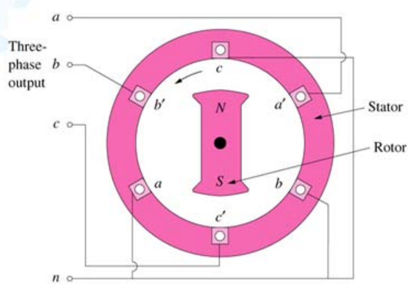
2. The instantaneous power in a three-phase system is constant
∴ There is less vibration in the rotating machinery which in turn performs more efficiently.

3. The amount of power loss in the three-phase system is only half the power loss in the cables for the single phase system.

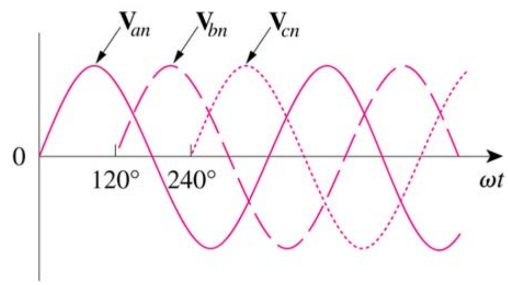
4. Thinner conductors can be used to transmit the same KVA at the same voltage.

Balanced Three – Phase Generator

A three-phase generator consists of a rotating magnet (rotor) surrounded by a stationary winding (stator).



A three-phase generator



The generated voltages

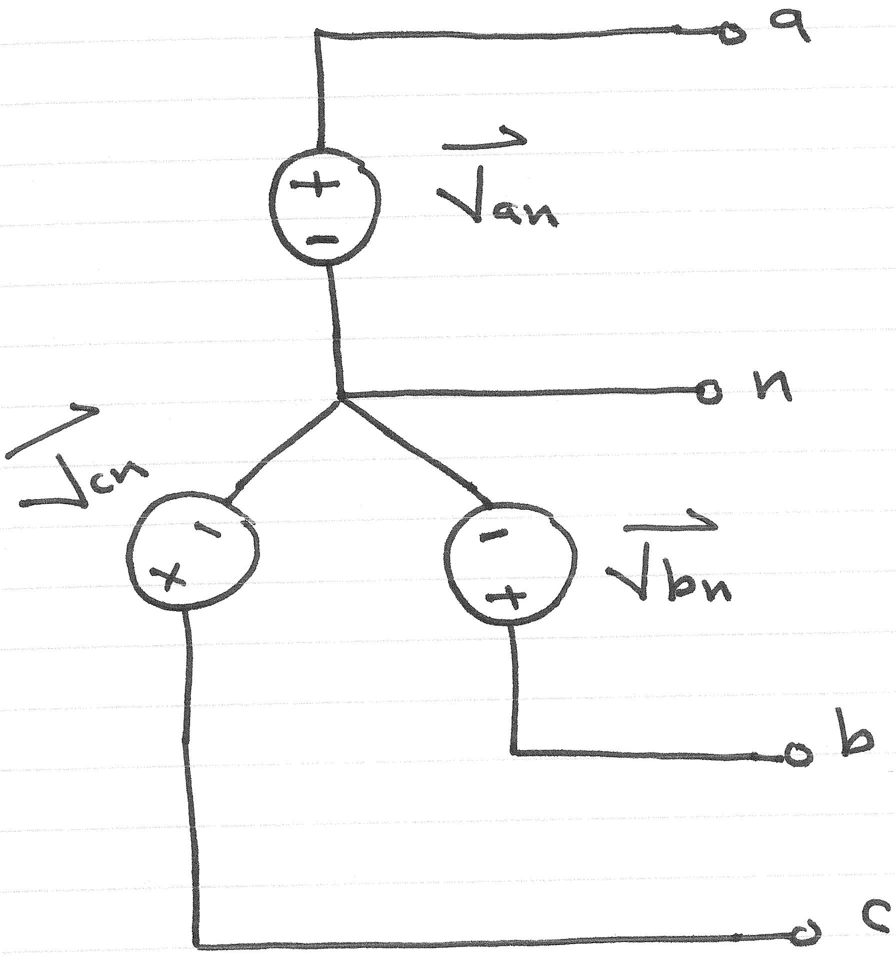
The Three – Phase Generator :

a) Has three induction coils.
b) Placed 120 a part on the rotor.
c) The three coils have an equal number of turns .
d) The voltage induced across each coil will have the same peak value, shape and frequency.

Balanced Three-phase Sources

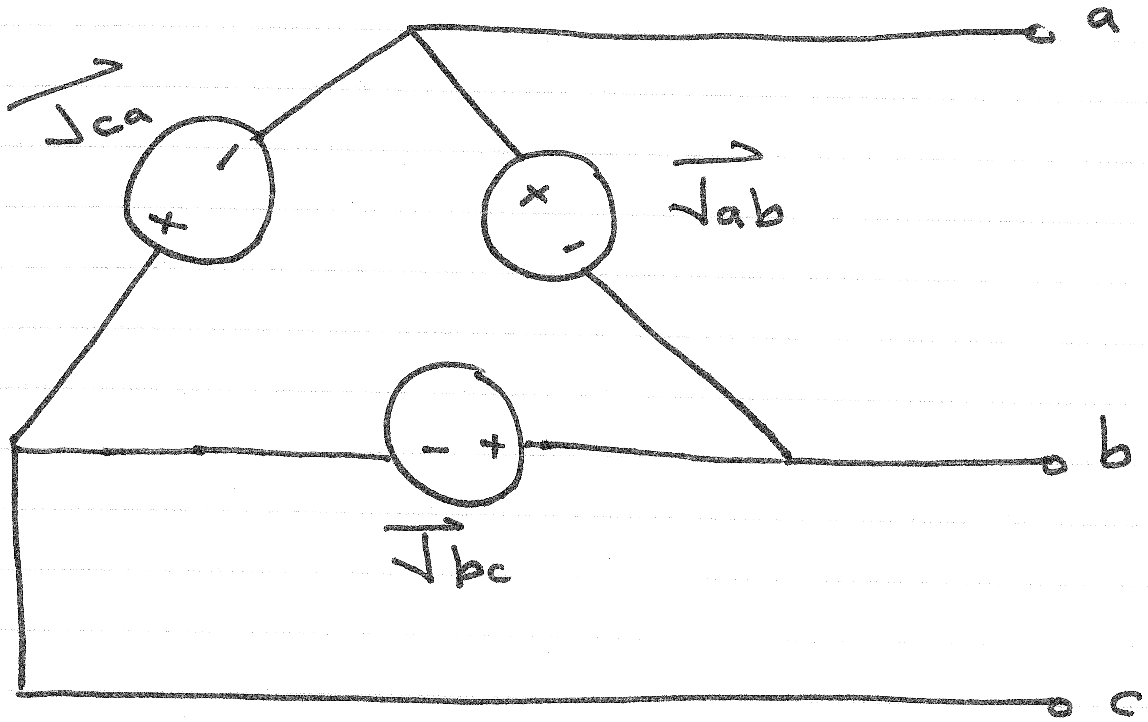
Two possible Configurations

1. The Y-connected Source



$\vec{V}_{an}$, $\vec{V}_{bn}$, and $\vec{V}_{cn}$ are Called the phase Voltages.

2. The Δ- Connected source



The phase Sequence

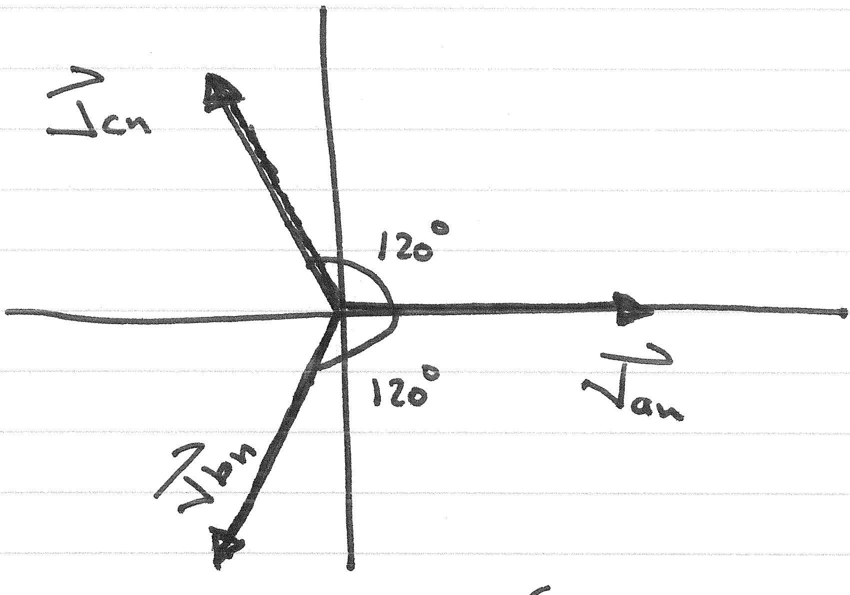
The phase sequence is the time order in which the voltages pass through their respective maximum values

1. abc sequence (positive sequence)

$$\vec{V}_{an} = V_p \angle 0^\circ$$

$$\vec{V}_{bn} = V_p \angle -120^\circ$$

$$\vec{V}_{cn} = V_p \angle +120^\circ$$

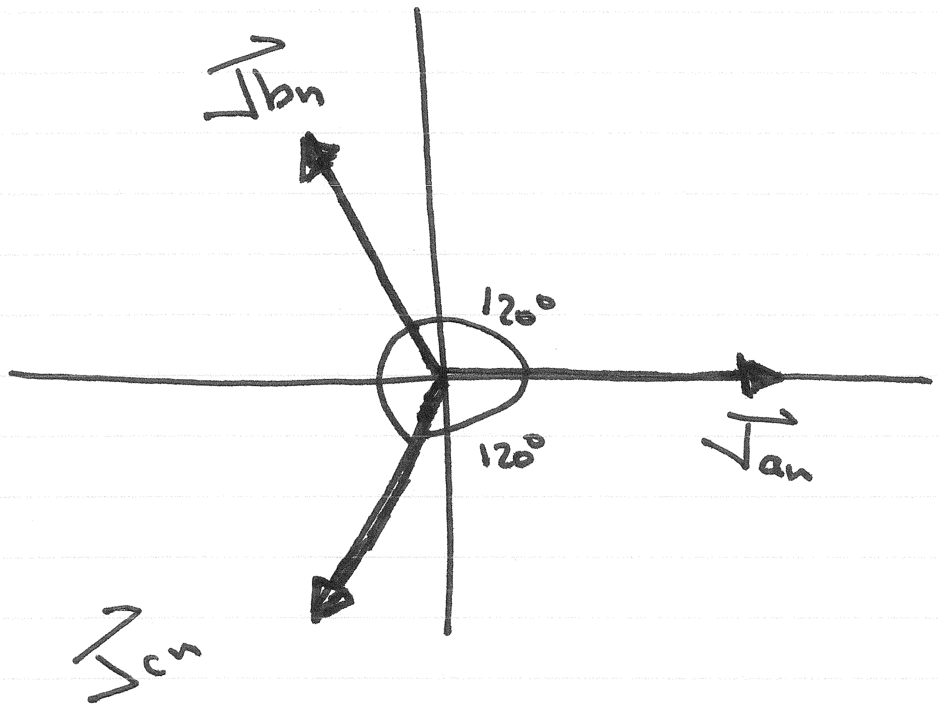


2. acb sequence (negative sequence)

$$\vec{V}_{an} = V_p \angle 0^\circ$$

$$\vec{V}_{bn} = V_p \angle +120^\circ$$

$$\vec{V}_{cn} = V_p \angle -120^\circ$$



$$\vec{V}_{an} + \vec{V}_{bn} + \vec{V}_{cn} = 0$$

$$\vec{V}_{an} = V_p \angle 0^\circ$$
$$\vec{V}_{bn} = V_p \angle -120^\circ$$
$$\vec{V}_{cn} = V_p \angle +120^\circ$$

$$\vec{V}_{an} = V_p$$
$$\vec{V}_{bn} = V_p \cos(-120^\circ) + j V_p \sin(-120^\circ)$$
$$\vec{V}_{bn} = V_p \left(-\frac{1}{2} - j\frac{\sqrt{3}}{2}\right)$$

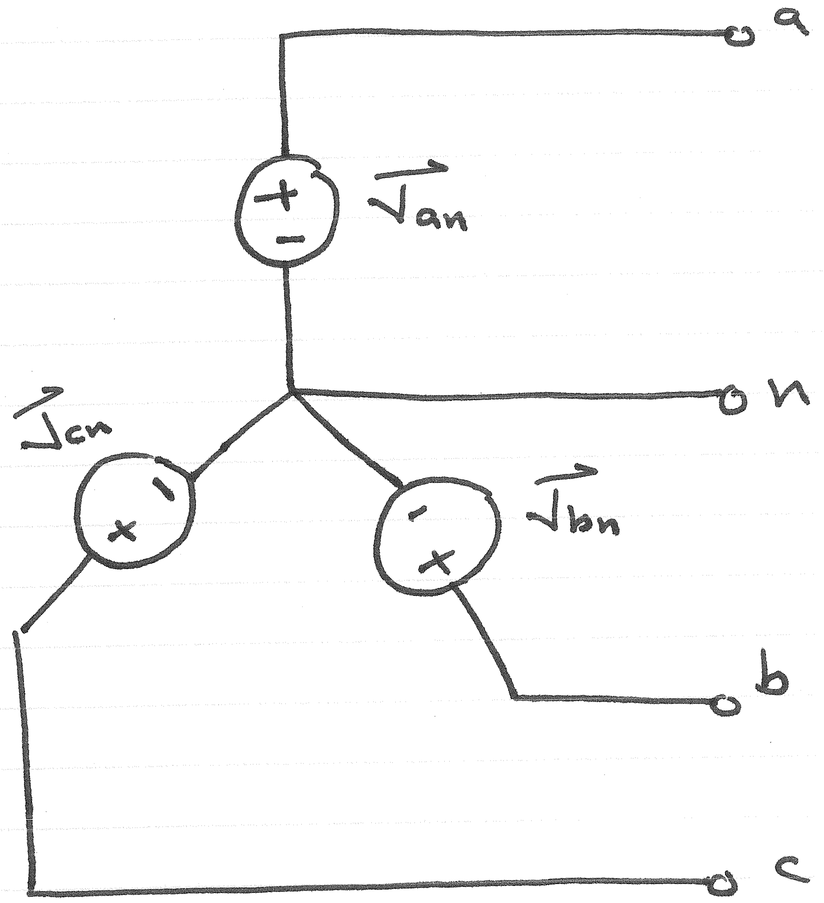
$$\vec{V}_{cn} = V_p \cos(+120^\circ) + j V_p \sin(+120^\circ)$$
$$\vec{V}_{cn} = V_p \left(-\frac{1}{2} + j\frac{\sqrt{3}}{2}\right)$$

$$\therefore \vec{V}_{an} + \vec{V}_{bn} + \vec{V}_{cn} = 0$$

$$\therefore v_{an}(t) + v_{bn}(t) + v_{cn}(t) = 0$$

Balanced Set

Line-to Line Voltages



$\vec{V}_{ab}$, $\vec{V}_{bc}$, $\vec{V}_{ca}$ are called the line-to-line voltages let $\vec{V}_{an} = V_p \angle 0^\circ$ ✓

$\vec{V}_{bn} = V_p \angle -120^\circ$ ✓

$\vec{V}_{cn} = V_p \angle +120^\circ$ ✓

$$\vec{V}_{ab} = \vec{V}_{an} + \vec{V}_{nb}$$

$$\vec{V}_{ab} = \vec{V}_{an} - \vec{V}_{bn}$$

$$\vec{V}_{ab} = V_p \angle 0^\circ - V_p \angle -120^\circ$$

$$\vec{V}_{ab} = V_p - V_p \left(\cos(-120^\circ) + j \sin(-120^\circ) \right)$$

$$\vec{V}_{ab} = V_p - V_p \left(-\frac{1}{2} - j \frac{\sqrt{3}}{2} \right)$$

$$\vec{V}_{ab} = V_p \left(1 + \frac{1}{2} + j \frac{\sqrt{3}}{2} \right)$$

$$\vec{V}_{ab} = V_p \left(\frac{3}{2} + j \frac{\sqrt{3}}{2} \right)$$

$$\vec{V}_{ab} = V_p \sqrt{\left(\frac{3}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} \angle \tan^{-1} \frac{\frac{\sqrt{3}}{2}}{3/2}$$

$\vec{V}_{ab} = V_p \sqrt{3} \angle +30^\circ$ ✓

$\vec{V}_{ab} = \sqrt{3}\, \vec{V}_{an} \angle +30^\circ$ ✓

$\vec{V}_{bc} = \sqrt{3}\, \vec{V}_{bn} \angle +30^\circ$ ✓

$\vec{V}_{ca} = \sqrt{3}\, \vec{V}_{cn} \angle +30^\circ$ ✓

For negative sequence :

$$\vec{V_{an}} = V_p \angle 0^\circ \quad V$$

$$\vec{V_{bn}} = V_p \angle +120^\circ \quad V$$

$$\vec{V_{cn}} = V_p \angle -120^\circ \quad V$$

$$\vec{V_{ab}} = V_p \sqrt{3} \angle -30^\circ \quad V$$

$$\therefore \vec{V_{ab}} = \sqrt{3}\, \vec{V_{an}} \angle -30^\circ \quad V$$

$$\therefore \vec{V_{bc}} = V_p \sqrt{3} \angle +90^\circ \quad V$$

$$\therefore \vec{V_{bc}} = \sqrt{3}\, \vec{V_{bn}} \angle -30^\circ \quad V$$

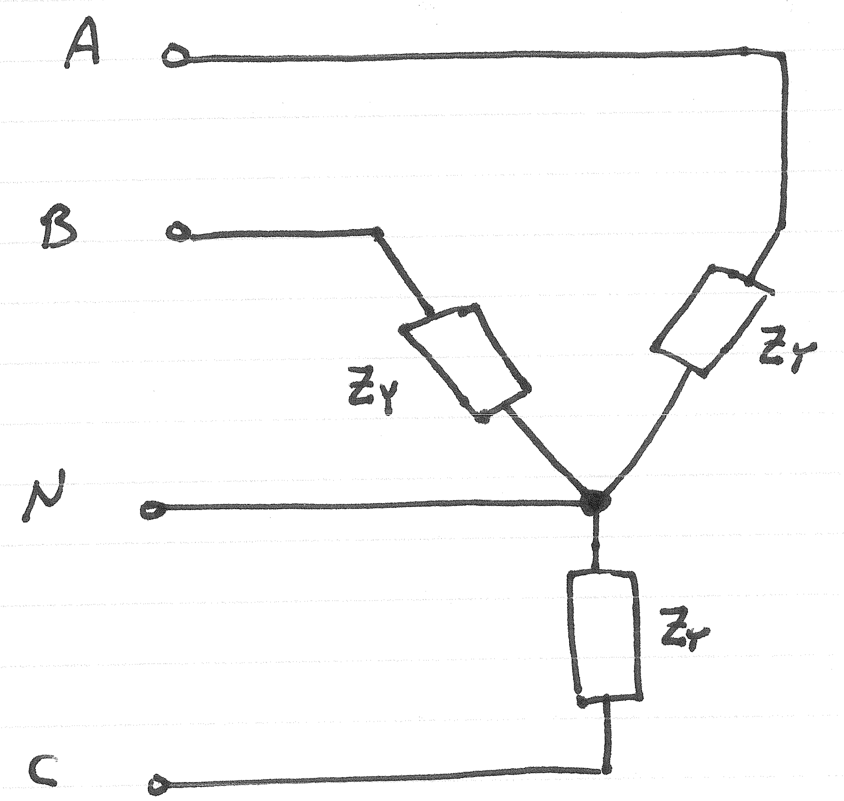
$$\therefore \vec{V_{ca}} = V_p \sqrt{3} \angle -150^\circ \quad V$$

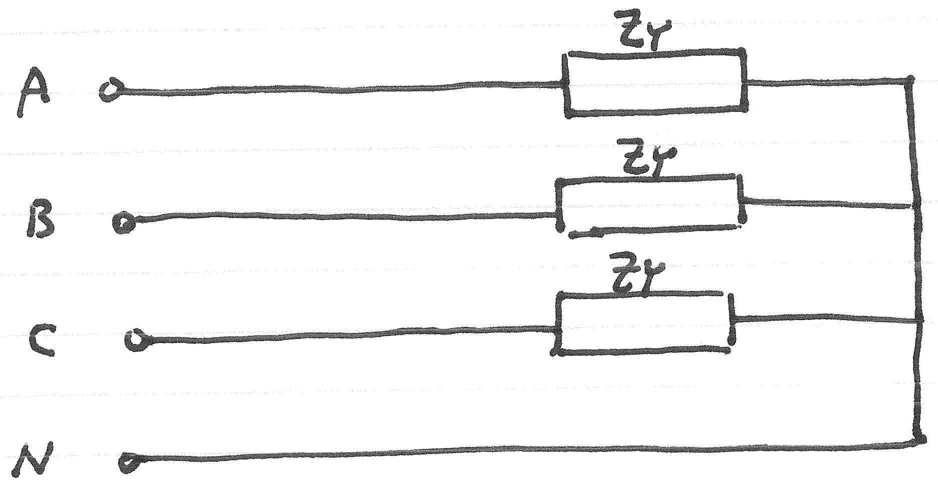
$$\therefore \vec{V_{ca}} = \sqrt{3}\, \vec{V_{cn}} \angle -30^\circ \quad V$$

Balanced Three phase Loads

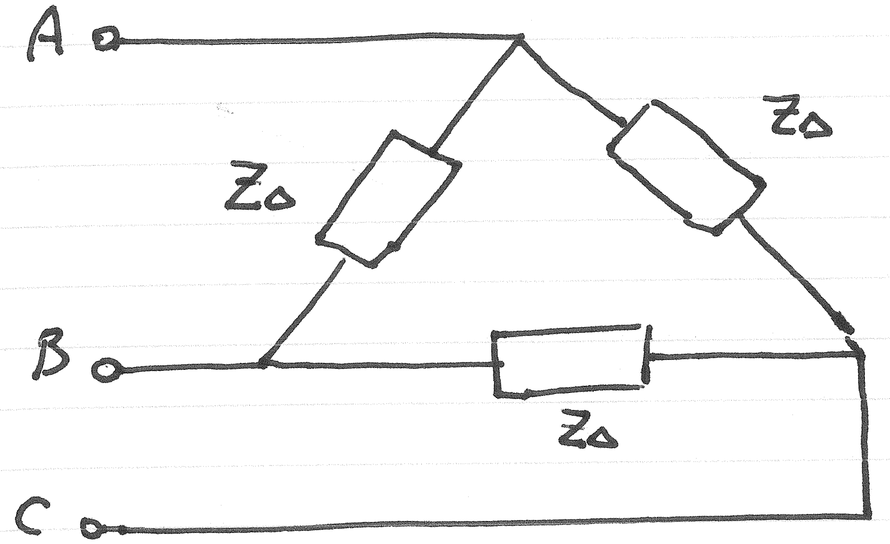
A balanced Load has equal impedances on all the phases.

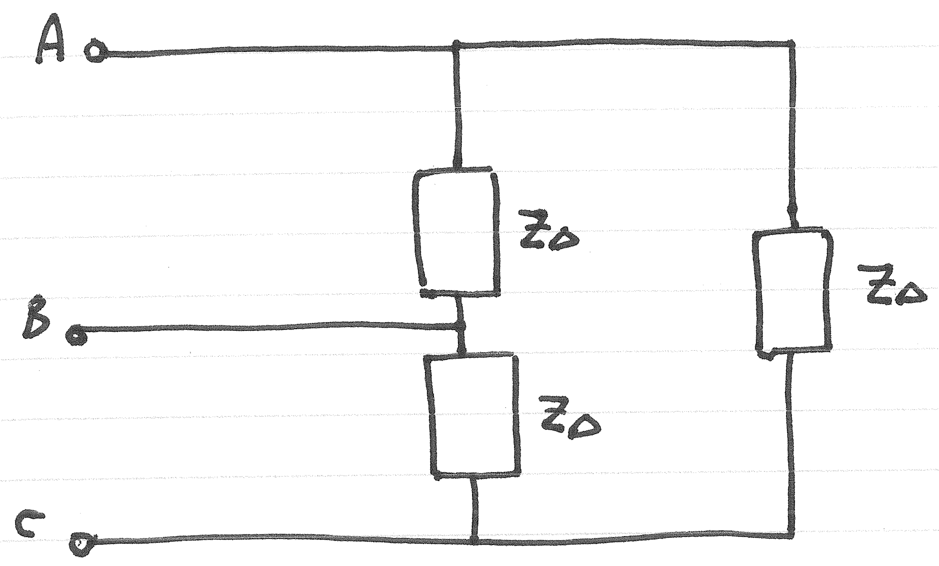
1) Y- Connected Load





2) Δ - Connected Load





$$Z_Y = \frac{Z_\Delta}{3}$$

$$Z_\Delta = 3 Z_Y$$

Three phase Connections

Both the three phase source and the three phase load Can be Connected either Wye or Delta

$\therefore$ We have 4 possible Connection types.

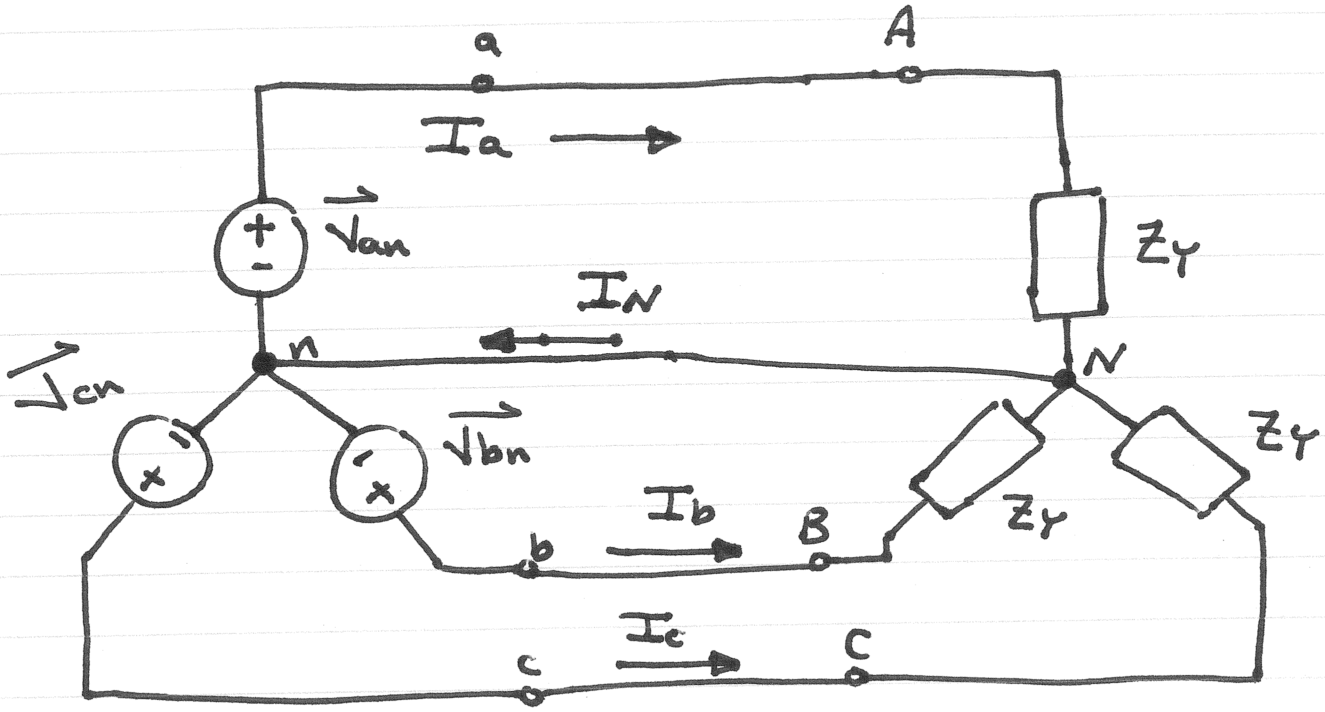
Y-Y Connection

Y-Δ Connection

Δ-Δ Connection

Δ-Y Connection

Balanced Y-Y system



$$\vec{I}_a = \frac{\vec{V}_{an}}{Z_Y}$$

$$\vec{I}_b = \frac{\vec{V}_{bn}}{Z_Y}$$

$$\vec{I}_c = \frac{\vec{V}_{cn}}{Z_Y}$$

KCL: $\vec{I}_N = \vec{I}_a + \vec{I}_b + \vec{I}_c$

$$= \frac{\vec{V}_{an}}{Z_Y} + \frac{\vec{V}_{bn}}{Z_Y} + \frac{\vec{V}_{cn}}{Z_Y}$$

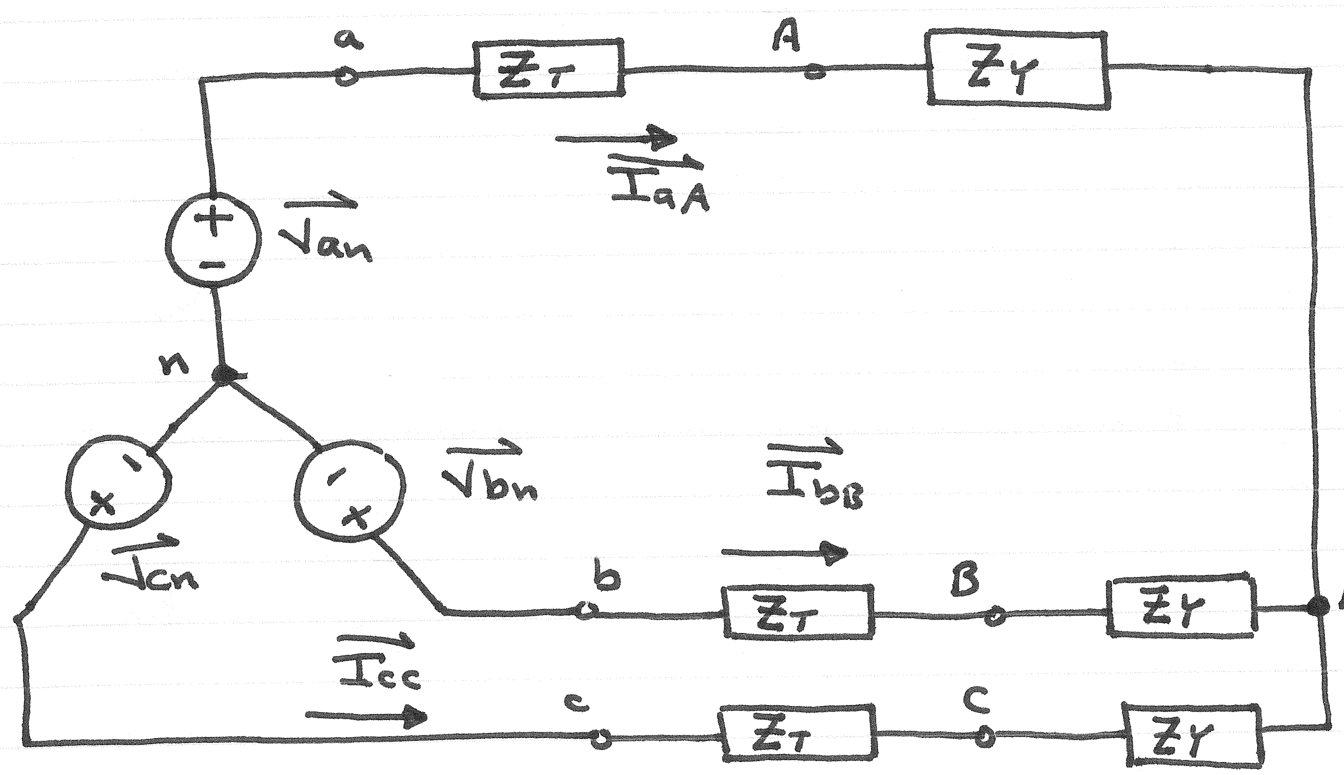
$$= \frac{1}{Z_Y}\left(\vec{V}_{an} + \vec{V}_{bn} + \vec{V}_{cn}\right)$$

$$\vec{I}_N = 0$$

$\therefore$ could be replaced by open Circuit

Example :

Calculate the Line currents.



$\vec{V}_{an} = 120\angle 0^\circ$ V rms

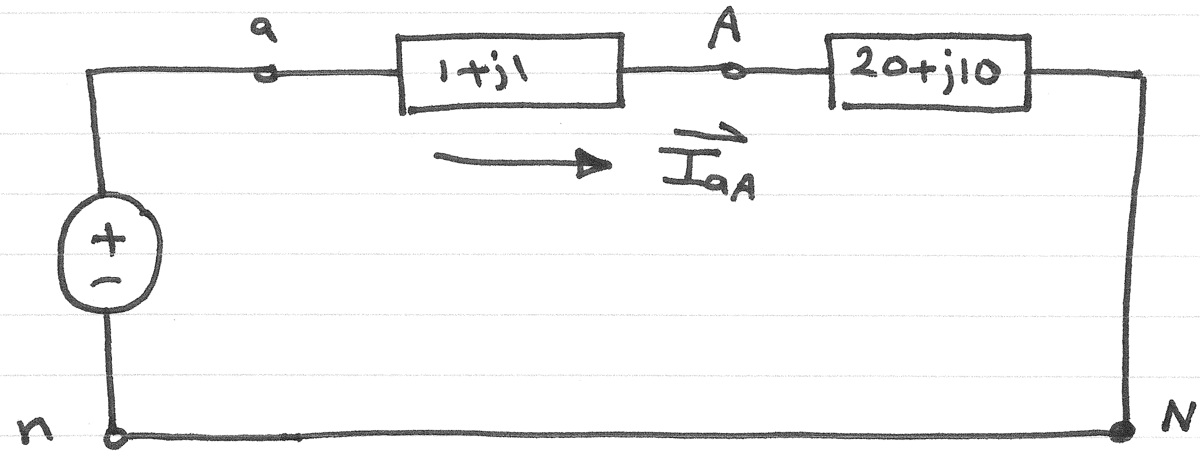
$\vec{V}_{bn} = 120\angle -120^\circ$ V rms

$\vec{V}_{cn} = 120\angle +120^\circ$ V rms

$Z_T = (1+j1)\ \Omega$

$Z_Y = (20+j10)\ \Omega$

Single phase representation



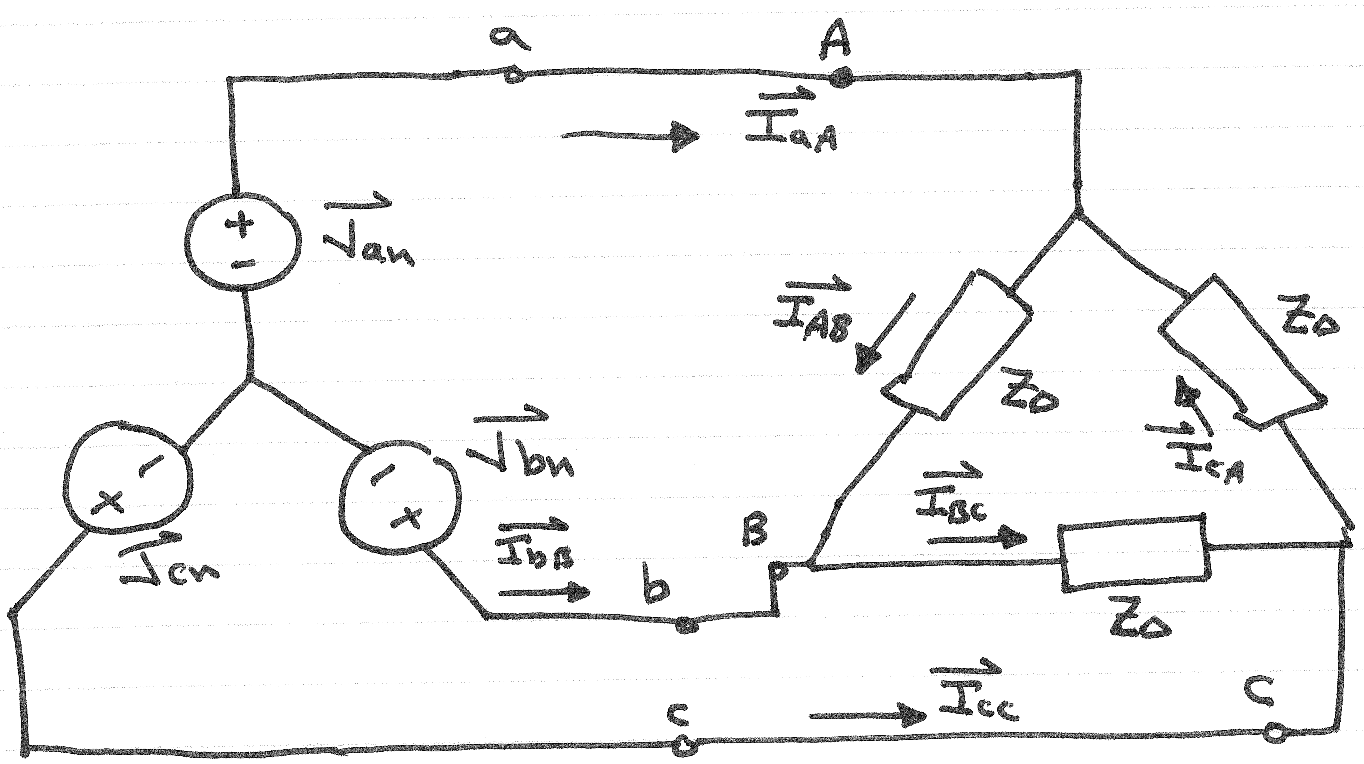
$$\overrightarrow{I_{aA}} = \frac{\overrightarrow{V_{an}}}{Z_T + Z_Y} = \frac{120 \angle 0^\circ}{21 + j\,11}$$

$$\therefore \quad \overrightarrow{I_{aA}} = 5.06 \angle -27.65^\circ \quad A \text{ rms}$$

$$\therefore \quad \overrightarrow{I_{bB}} = 5.06 \angle -147.65^\circ \quad A \text{ rms}$$

$$\therefore \quad \overrightarrow{I_{cC}} = 5.06 \angle 92.35^\circ \quad A \text{ rms}$$

Balanced Y-Δ System



Example:

$\vec{V}_{an} = 120\angle 30^\circ$ V_{rms}

$Z_\Delta = (6+j6)\ \Omega$

positive sequence

Calculate the Line Currents.

$\vec{V}_{ab} = \vec{V}_{AB} = 120\sqrt{3}\angle 60^\circ$ V_{rms}

$\vec{I}_{AB} = \dfrac{\vec{V}_{AB}}{Z_\Delta} = 24.5\angle 15^\circ$ A_{rms}

$\therefore\ \vec{I}_{BC} = 24.5\angle -105^\circ$ A_{rms}

$\therefore\ \vec{I}_{CA} = 24.5\angle 135^\circ$ A_{rms}

$\overrightarrow{I_{AB}}$, $\overrightarrow{I_{BC}}$, and $\overrightarrow{I_{CA}}$ are the phase currents of the Load.

KCL :

$$\overrightarrow{I_{aA}} = \overrightarrow{I_{AB}} - \overrightarrow{I_{CA}}$$

$$\overrightarrow{I_{aA}} = 24.5\angle 15^\circ - 24.5\angle 135^\circ$$

$$\overrightarrow{I_{aA}} = 42.44\angle -15^\circ \ A_{rms}$$

$$\overrightarrow{I_{aA}} = \sqrt{3}\ \overrightarrow{I_{AB}}\angle -30^\circ$$

Line current Lags the phase current by 30° only for abc sequence

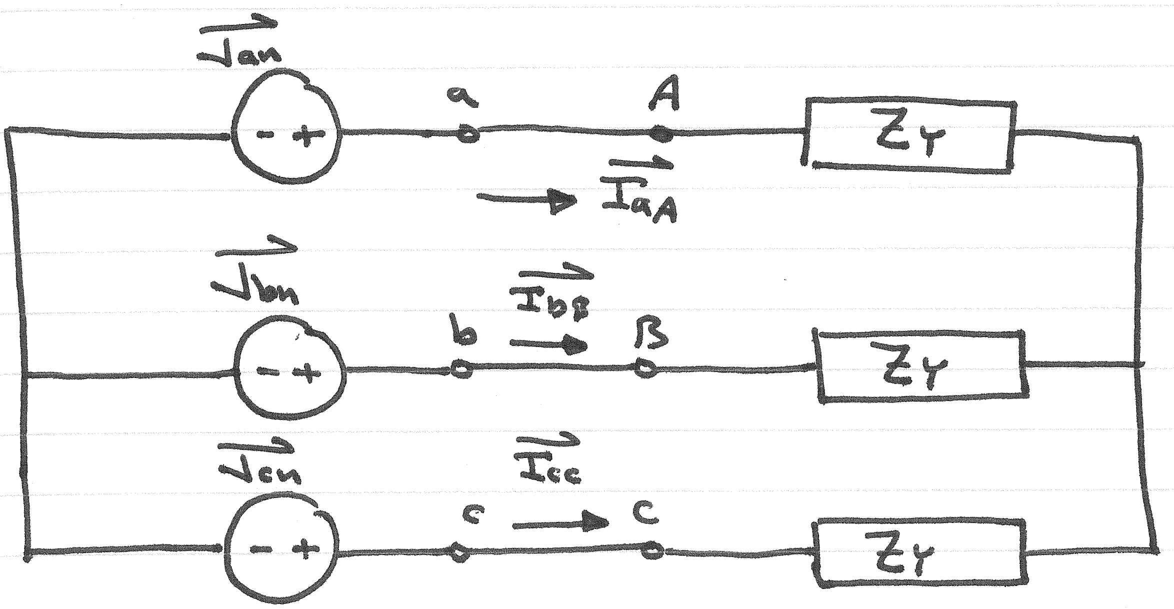
$$\therefore\ \overrightarrow{I_{bB}} = 42.44\angle -135^\circ \ A_{rms}$$

$$\therefore\ \overrightarrow{I_{cC}} = 42.44\angle 105^\circ \ A_{rms}$$

Second method

Using $\Delta - Y$ Transformation

$$Z_Y = \frac{Z_\Delta}{3}$$



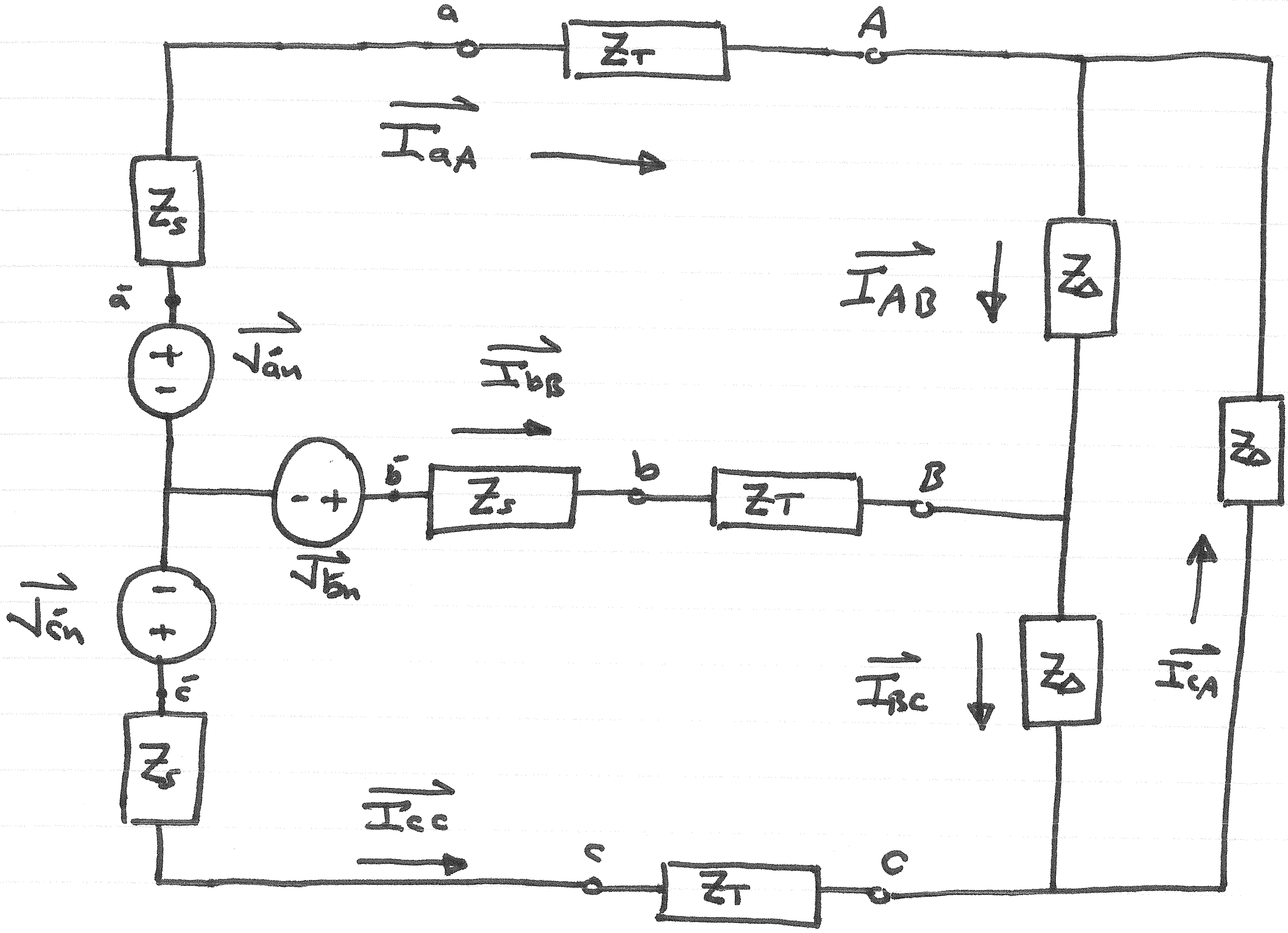
$$Z_Y = \frac{6 + j6}{3} = (2 + j2)\ \Omega$$

$$\vec{I}_{aA} = \frac{\vec{V}_{an}}{Z_Y} = 42.44 \angle -15^\circ \ \text{A}_{rms}$$

$$\therefore \vec{I}_{bB} = 42.44 \angle -135^\circ \ \text{A}_{rms}$$

$$\therefore \vec{I}_{cC} = 42.44 \angle 105^\circ \ \text{A}_{rms}$$

Example



given $\vec{V}_{a'n} = 120\angle 0°$ V_{rms}

abc sequence

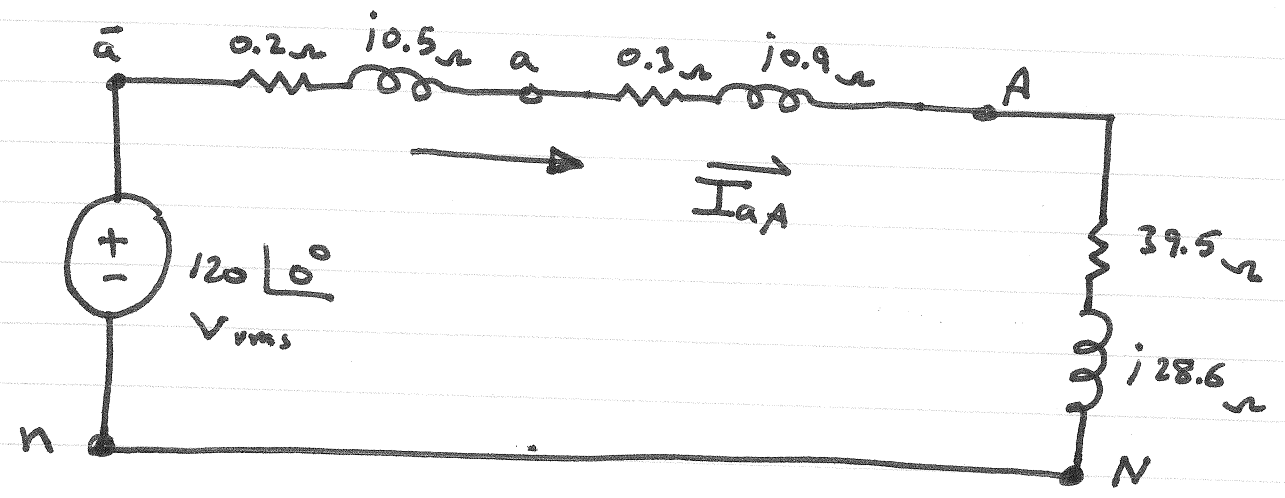
$Z_s = (0.2 + j0.5)\,\Omega$

$Z_T = (0.3 + j0.9)\,\Omega$

$Z_\Delta = (118.5 + j85.8)\,\Omega$

1) Calculate the line currents

Single phase representation



$$Z_Y = \frac{Z_\Delta}{3} = \frac{118.5+j85.8}{3} = (39.5+j\,28.6)\,\Omega$$

$$\overrightarrow{I_{aA}} = \frac{120\angle 0^\circ}{(0.2+j0.5)+(0.3+j0.9)+39.5+j\,28.6}$$

$$\overrightarrow{I_{aA}} = 2.4\angle -36.87^\circ \quad A_{rms}$$

$$\therefore \overrightarrow{I_{bB}} = 2.4\angle -156.87^\circ \quad A_{rms}$$

$$\therefore \overrightarrow{I_{cC}} = 2.4\angle 83.13^\circ \quad A_{rms}$$

2) Calculate the phase currents of the load

$$\overrightarrow{I_{AB}} = \frac{1}{\sqrt{3}} \angle{+30^\circ} \; \overrightarrow{I_{aA}}$$

$$\therefore \quad \overrightarrow{I_{AB}} = 1.39 \angle{-6.87^\circ} \quad A_{rms}$$

$$\therefore \quad \overrightarrow{I_{BC}} = 1.39 \angle{-126.87^\circ} \quad A_{rms}$$

$$\therefore \quad \overrightarrow{I_{CA}} = 1.39 \angle{113.13^\circ} \quad A_{rms}$$

3) Calculate the phase voltages at the load terminals ; $\overrightarrow{V_{AB}}$, $\overrightarrow{V_{BC}}$, and $\overrightarrow{V_{CA}}$

a) First method

$$\overrightarrow{V_{AB}} = Z_\Delta \; \overrightarrow{I_{AB}}$$

$$\overrightarrow{V_{AB}} = (118.5 + j\,85.8)\,(1.39 \angle{-6.87^\circ})$$

$$\overrightarrow{V_{AB}} = 202.72 \angle{29.04^\circ} \quad V_{rms}$$

$$\therefore \quad \overrightarrow{V_{BC}} = 202.72 \angle{-90.96^\circ} \quad V_{rms}$$

$$\therefore \quad \overrightarrow{V_{CA}} = 202.72 \angle{149.04^\circ} \quad V_{rms}$$

b) second method

From the single phase representation

$$\vec{V}_{AN} = Z_Y \vec{I}_{aA}$$

$$\vec{V}_{AN} = (39.5 + j\,28.6)\,(2.4 \angle -36.87^\circ)$$

$$\vec{V}_{AN} = 117.04 \angle -0.96^\circ \quad \text{V rms}$$

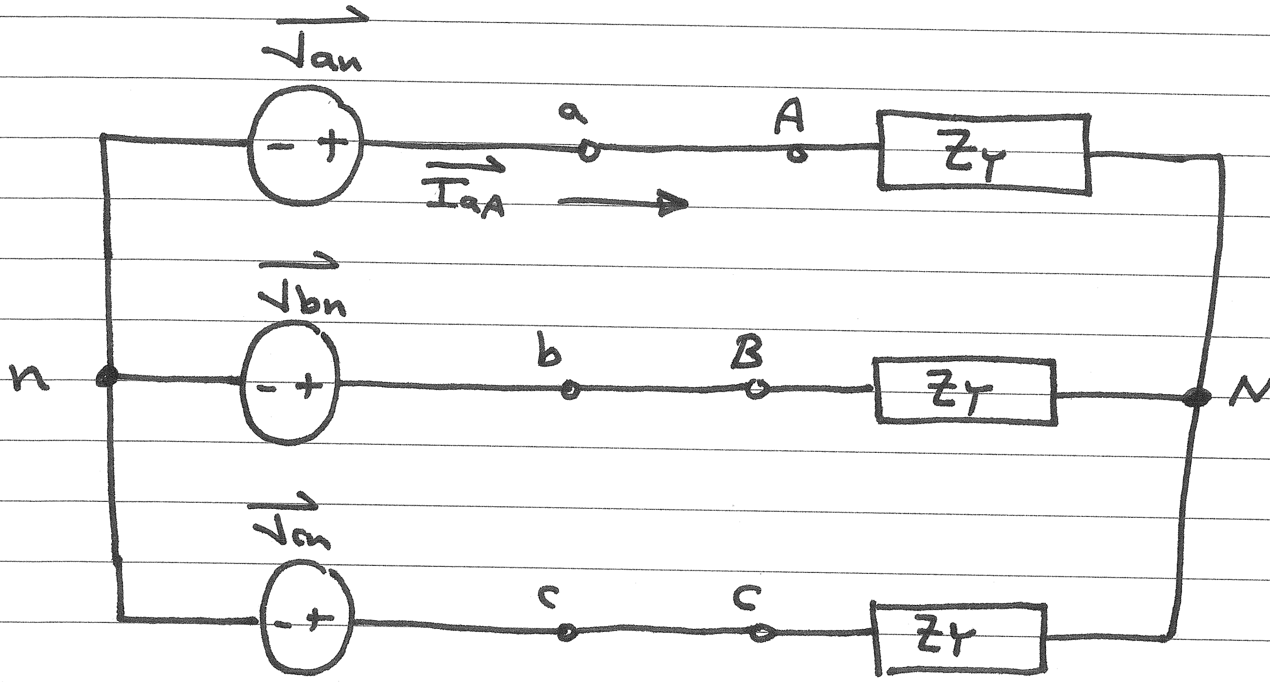
$$\therefore \quad \vec{V}_{AB} = \sqrt{3} \angle +30^\circ \; \vec{V}_{AN}$$

$$\therefore \quad \vec{V}_{AB} = 202.72 \angle 29.04^\circ \quad \text{V rms}$$

$$\therefore \quad \vec{V}_{BC} = 202.72 \angle -90.96^\circ \quad \text{V rms}$$

$$\therefore \quad \vec{V}_{CA} = 202.72 \angle 149.04^\circ \quad \text{V rms}$$

Power in a Balanced System



* The total instantaneous power in a balanced three phase system is Constant

$$v_{AN}(t) = \sqrt{2}\, V_p \cos \omega t$$

$$v_{BN}(t) = \sqrt{2}\, V_p \cos(\omega t - 120^\circ)$$

$$v_{CN}(t) = \sqrt{2}\, V_p \cos(\omega t + 120^\circ)$$

$$i_{aA}(t) = \sqrt{2}\, I_p \cos(\omega t - \theta)$$

$$i_{bB}(t) = \sqrt{2}\, I_p \cos(\omega t - \theta - 120^\circ)$$

$$i_{cC}(t) = \sqrt{2}\, I_p \cos(\omega t - \theta + 120^\circ)$$

$$P(t) = P_a(t) + P_b(t) + P_c(t)$$

$$P_a(t) = 2 V_p I_p \cos \omega t \cos(\omega t - \theta)$$

$$P_b(t) = 2 V_p I_p \cos(\omega t - 120^\circ) \cos(\omega t - \theta - 120^\circ)$$

$$P_c(t) = 2 V_p I_p \cos(\omega t + 120^\circ) \cos(\omega t - \theta + 120^\circ)$$

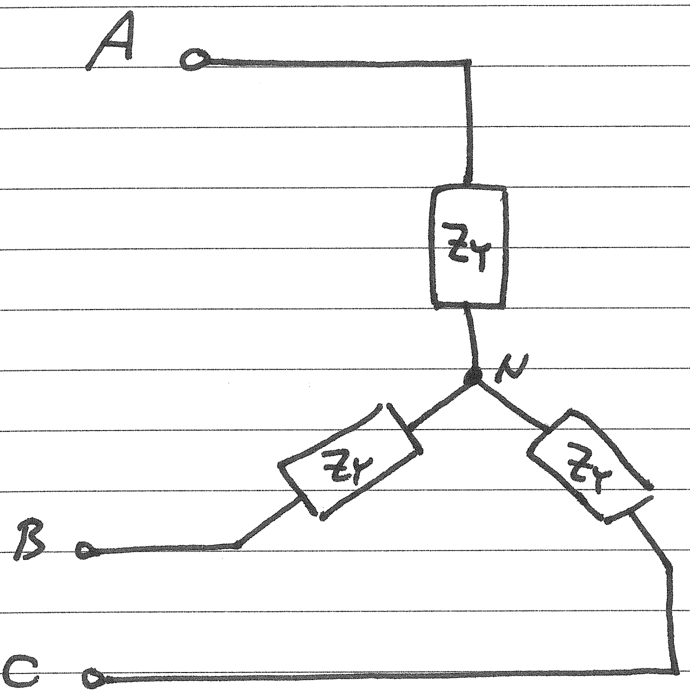
using $\cos A \cos B = \frac{1}{2}\left[\cos(A+B) + \cos(A-B)\right]$

$$P(t) = V_p I_p \left[3 \cos \theta\right]$$

$$P(t) = 3 V_p I_p \cos \theta$$

Power Calculation in a Balanced 3 ϕ systems

1) Average power in Balanced Y-Load



$$P_A = V_{AN} I_{AN} \cos(\theta_{vA} - \phi_{iA})$$

$$P_B = V_{BN} I_{BN} \cos(\theta_{vB} - \phi_{iB})$$

$$P_C = V_{CN} I_{CN} \cos(\theta_{vc} - \phi_{ic})$$

$$V_{AN} = V_{BN} = V_{CN} = V\phi$$

$$I_{AN} = I_{BN} = I_{CN} = I\phi$$

$$\theta_{vA} - \phi_{iA} = \theta_{vB} - \phi_{iB} = \theta_{vc} - \phi_{ic}$$

$$\therefore \; P_A = P_C = P_C = V_\phi I_\phi \cos\theta$$

$$P_T = 3 V_\phi I_\phi \cos\theta$$

But $V_\phi = \dfrac{V_L}{\sqrt{3}}$

$$I_\phi = I_L$$

$$\therefore \; P_T = \sqrt{3}\, V_L I_L \cos\theta$$

2) Reactive Power in Balanced Y- load

$$Q_A = Q_B = Q_C = V_\phi I_\phi \sin\theta$$

$$Q_T = 3 V_\phi I_\phi \sin\theta$$

$$Q_T = \sqrt{3}\, V_L I_L \sin\theta$$

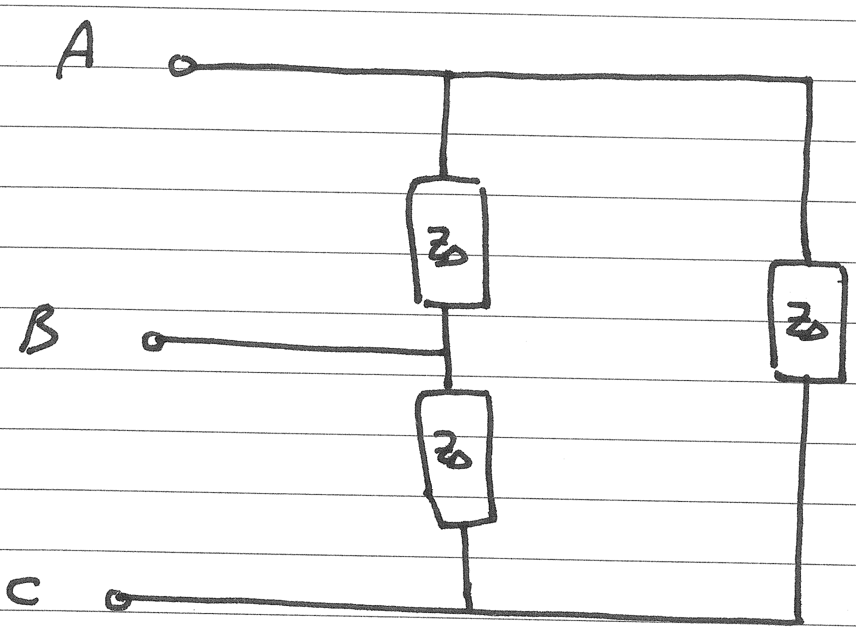
3) Complex Power in Balanced Y- load

$$\vec{S_A} = \vec{S_B} = \vec{S_C} = \vec{V_\phi} \cdot \vec{I_\phi^*} = P_\phi + j Q_\phi$$

$$\vec{S_T} = 3 \vec{S_\phi} = 3 \vec{V_\phi} \cdot \vec{I_\phi^*}$$

$$\vec{S_T} = \sqrt{3}\, V_L I_L \angle\theta$$

Power Calculation in Balanced Δ - load



$$P_A = V_{AB}\, I_{AB} \cos\left(\theta_{v_{AB}} - \phi_{i_{AB}}\right)$$

$$P_B = V_{BC}\, I_{BC} \cos\left(\theta_{v_{BC}} - \phi_{i_{BC}}\right)$$

$$P_C = V_{CA}\, I_{CA} \cos\left(\theta_{v_{CA}} - \phi_{i_{CA}}\right)$$

$$V_{AB} = V_{BC} = V_{CA}$$

$$I_{AB} = I_{BC} = I_{CA}$$

$$\theta_{v_{AB}} - \phi_{i_{AB}} = \theta_{v_{BC}} - \phi_{i_{BC}} = \theta_{v_{CA}} - \phi_{i_{CA}}$$

$$\therefore \quad P_A = P_B = P_C = P_\phi = V_\phi\, I_\phi \cos\theta$$

$$P_T = 3\,P_\phi = 3\,V_\phi\, I_\phi \cos\theta$$

$$V_\phi = V_L$$

$$I_\phi = \frac{I_L}{\sqrt{3}}$$

$$\therefore\ P_T = \sqrt{3}\ V_L\, I_L \cos\theta$$

$$Q_\phi = V_\phi\, I_\phi \sin\theta$$

$$Q_T = 3\,Q_\phi = 3\,V_\phi\, I_\phi \sin\theta$$

$$Q_T = \sqrt{3}\ V_L\, I_L \sin\theta$$

$$\overrightarrow{S_T} = 3\,\overrightarrow{S_\phi}$$

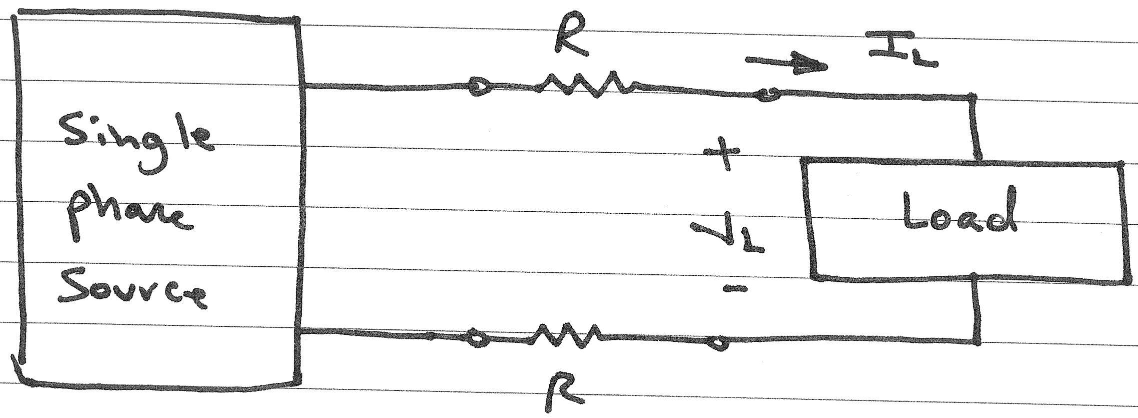
$$\overrightarrow{S_\phi} = \overrightarrow{V_\phi}\,\overrightarrow{I_\phi^*} = P_\phi + j\,Q_\phi$$

$$\overrightarrow{S_T} = P_T + j\,Q_T$$

$$\overrightarrow{S_T} = \sqrt{3}\ V_L\, I_L \angle\theta$$

Comparing the Power Loss

a) a single phase system

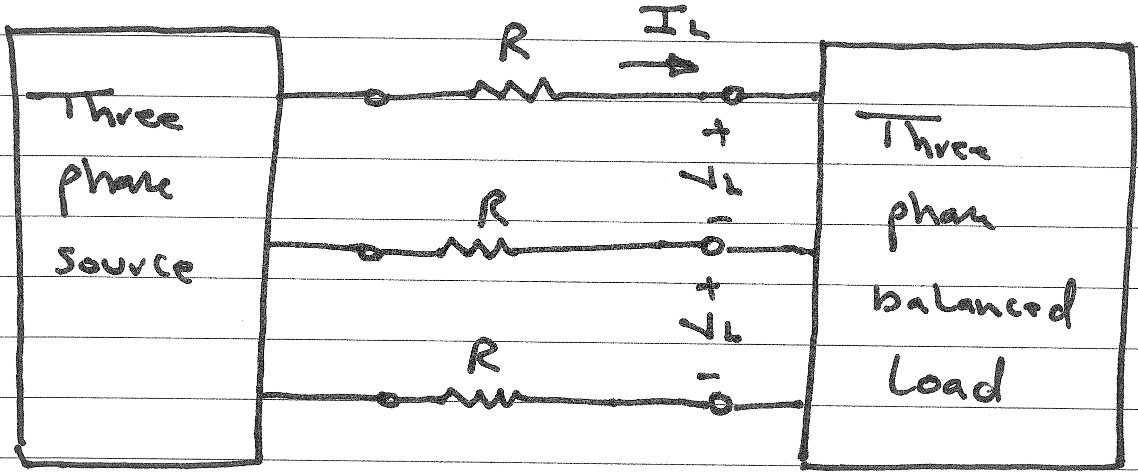


$$P_{Loss} = 2 I_L^2 \cdot R$$

$$I_L = \frac{P_L}{V_L \cdot pf}$$

$$P_{Loss} = 2 \frac{P_L^2}{V_L^2 \cdot pf^2} R$$

b) a three phase system



$$P_{Loss} = 3\ I_L^2 \cdot R$$

$$I_L = \frac{P_L}{\sqrt{3}\ V_L \cdot Pf}$$

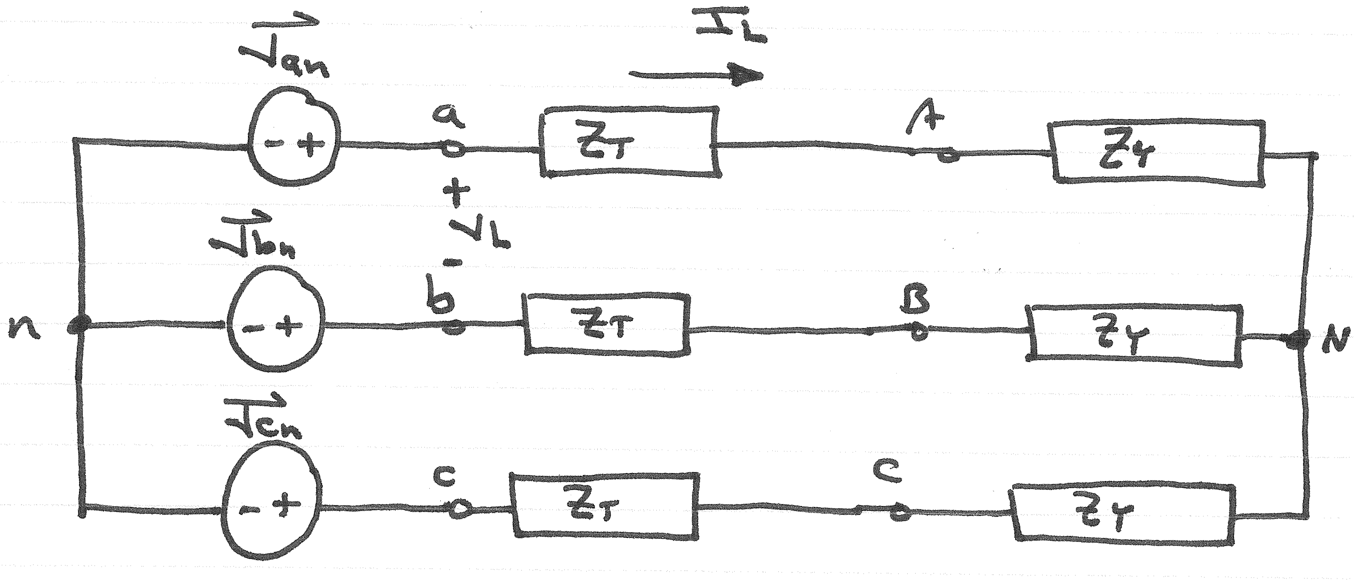
$$P_{Loss} = \frac{P_L^2}{V_L^2 \cdot Pf^2} \cdot R$$

Example

A balanced 3ϕ load requires 480 KW at a Lagging power factor of 0.8. The Load is fed from a Line having an impedance of $(0.005 + j0.025)\ \Omega/\phi$

The Line voltage at the terminal of the load is 600 V rms

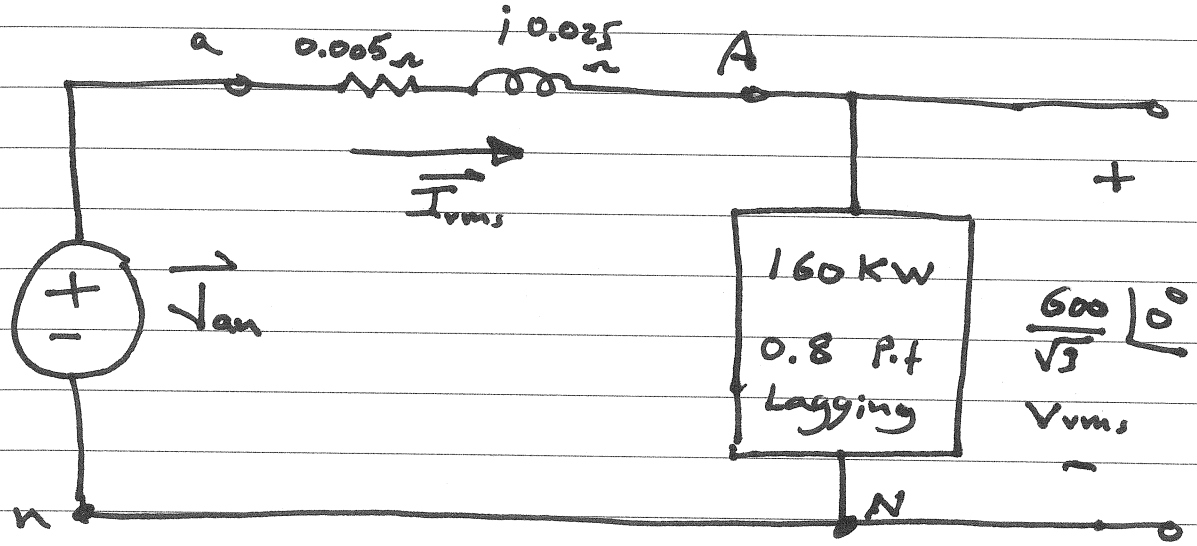
1) Calculate the magnitude of the line Current
2) Calculate the magnitude of the Line voltage at the sending end of the Line
3) Calculate the power factor at the sending end of the Line.



$\vec{V}_{an}$
I_L
a
Z_T
A
Z_Y
+
V_L
-
$\vec{V}_{bn}$
n
b
Z_T
B
Z_Y
N
$\vec{V}_{cn}$
c
Z_T
C
Z_Y

Solution

Single phase representation



1) $P_{av} = 160$ KW

$Q = P_{av} \tan \cos^{-1} P.f = 120$ KVAR

$\vec{S} = P_{av} + jQ$

$\vec{S} = 160 + j\,120$ KVA

$\vec{S} = \vec{V}_{rms} \cdot \vec{I}^{*}_{rms}$

$\therefore \vec{I}_{rms} = 577.35 \angle -36.87^\circ \; A_{rms}$

$\therefore I_L = 577.35 \; A_{rms}$

2) $\vec{V}_{an} = (0.005 + j0.025)\,\vec{I}_{rms} + \frac{600}{\sqrt{3}} \angle 0^\circ$

$\vec{V}_{an} = 357.51 \angle 1.57^\circ \; V_{rms}$

$\therefore \quad V_{an} = 357.51 \quad V_{rms}$

$\therefore \quad V_L = \sqrt{3} \; V_{an}$

$\therefore \quad V_L = 619.23 \quad V_{rms}$

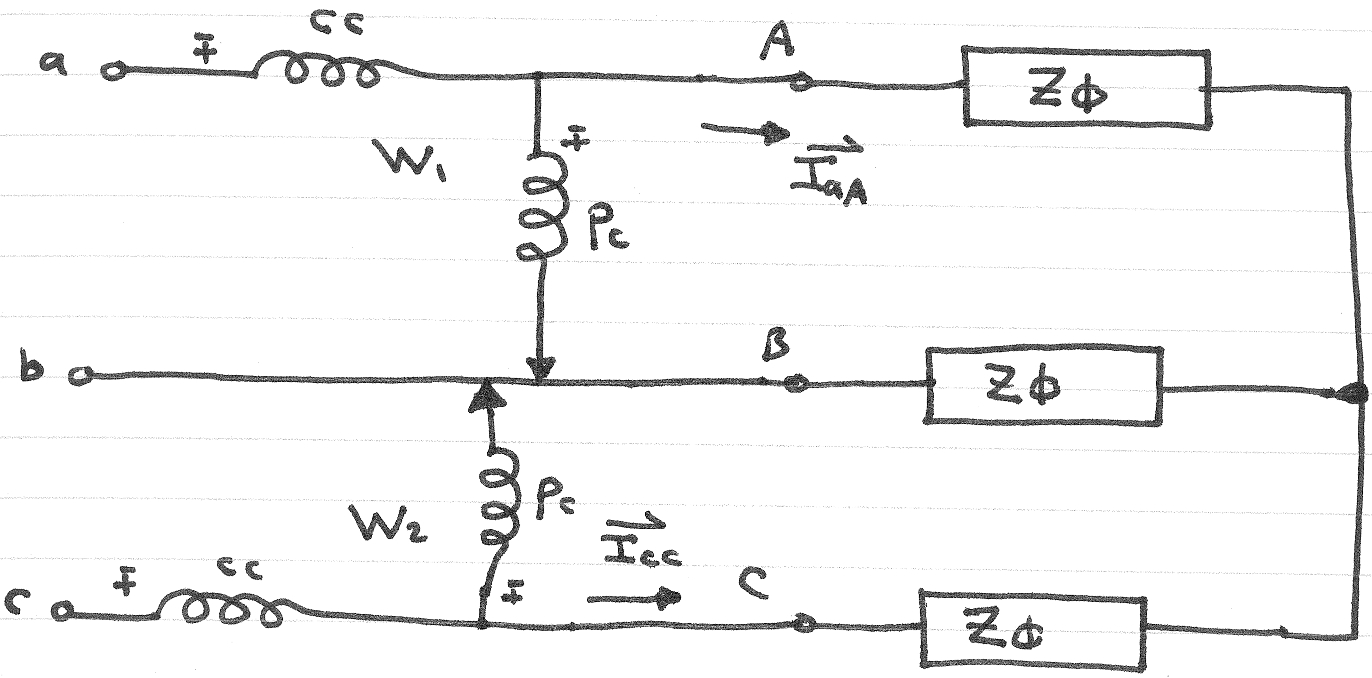
3) $Pf = \cos(\theta_v - \phi_i)$

$Pf = \cos(1.57^\circ + 36.87^\circ)$

$Pf = 0.783$ Lagging

Measuring Average power in 3ϕ system

The Two-Wattmeter method



$Z\phi = |Z| \angle \theta_z$; θ_z = impedance angle

$$W_1 = V_{AB}\, I_{aA} \cos \theta_1$$

θ_1 = The angle between $\overrightarrow{V_{AB}}$ and $\overrightarrow{I_{aA}}$

$$\theta_1 = \theta_z + 30^\circ$$

$$\overrightarrow{V_{AB}} = \overrightarrow{V_{AN}} \sqrt{3} \angle 30^\circ$$

$$\overrightarrow{I_{aA}} = \overrightarrow{I_{AN}}$$

$$\therefore \theta_1 = \theta_z + 30^\circ$$

$$\therefore W_1 = V_L\, I_L \cos(\theta_z + 30^\circ)$$

$$W_2 = V_{CB}\, I_{CC} \cos\theta_2$$

θ_2 = The angle between $\vec{V}_{CB}$ and $\vec{I}_{CC}$

$$\theta_2 = \theta_Z - 30^\circ$$

$$\vec{V}_{CB} = -\vec{V}_{BC}$$

$$\vec{V}_{CB} = \vec{V}_{BC} \angle 180^\circ$$

$$\vec{V}_{CB} = \vec{V}_{CA} \angle -240^\circ \angle 180^\circ$$

$$\vec{V}_{CB} = \vec{V}_{CA} \angle -60^\circ$$

$$\vec{V}_{CB} = \sqrt{3}\, \vec{V}_{CN} \angle +30^\circ \angle -60^\circ$$

$$\vec{V}_{CB} = \sqrt{3}\, \vec{V}_{CN} \angle -30^\circ$$

$$\vec{I}_{CC} = \vec{I}_{CN}$$

$$\therefore\ \theta_2 = \theta_Z - 30^\circ$$

$$\therefore\ W_2 = V_L\, I_L \cos\left(\theta_Z - 30^\circ\right)$$

$$W_1 = V_L I_L \cos(\theta_z + 30^\circ)$$

$$W_2 = V_L I_L \cos(\theta_z - 30^\circ)$$

$$P_T = W_1 + W_2$$

$$\cos(\theta_z + 30^\circ) = \cos\theta_z \cos 30^\circ - \sin\theta_z \sin 30^\circ$$

$$\cos(\theta_z - 30^\circ) = \cos\theta_z \cos 30^\circ + \sin\theta_z \sin 30^\circ$$

$$\therefore \quad W_1 + W_2 = V_L I_L \left(2 \cos\theta_z \cos 30^\circ\right)$$

$$W_1 + W_2 = \sqrt{3}\, V_L I_L \cos\theta_z$$

Example

Calculate the reading of each Wattmeter if the phase voltage at the load is $120\angle 0^\circ$ Vrms

a) and $Z_\phi = (8+j6)\ \Omega$

$$\vec{I}_{aA} = \frac{\vec{V}_{AN}}{Z_\phi} = 12\angle -36.87^\circ \ A_{rms}$$

$$Z_\phi = 8+j6 = 10\angle 36.87^\circ \ \Omega$$

$$V_L = \sqrt{3}\cdot(120) \ V_{rms}$$

$$I_L = 12 \ A_{rms}$$

$$\theta_z = 36.87^\circ$$

$$W_1 = V_L I_L \cos(\theta_z + 30^\circ)$$

$$= (120\sqrt{3})(12)\cos(66.87^\circ)$$

$$W_1 = 979.75 \ \text{Watt}$$

$$W_2 = V_L I_L \cos(\theta_z - 30^\circ)$$

$$W_2 = (120\sqrt{3})(12)\cos(6.87^\circ)$$

$$W_2 = 2476.25 \ \text{Watt}$$

b) $Z\phi = 8 - j6 = 10\underline{|-36.87^\circ}\ \Omega$

$\therefore \theta_z = -36.87^\circ$

$W_1 = (120\sqrt{3})(12)\cos(-36.87^\circ + 30^\circ)$

$W_2 = 2476.25$ Watt

$W_2 = (120\sqrt{3})(12)\cos(-36.87^\circ - 30^\circ)$

$W_2 = 979.75$ Watt

c) $Z\phi = 5 + j5\sqrt{3} = 10\underline{|60^\circ}$

$\therefore \theta_z = 60^\circ$

$W_1 = (120\sqrt{3})(12)\cos(60 + 30)$

$W = 0$

$W_2 = (120\sqrt{3})(12)\cos(60 - 30^\circ)$

$W_2 = 2160$ Watt

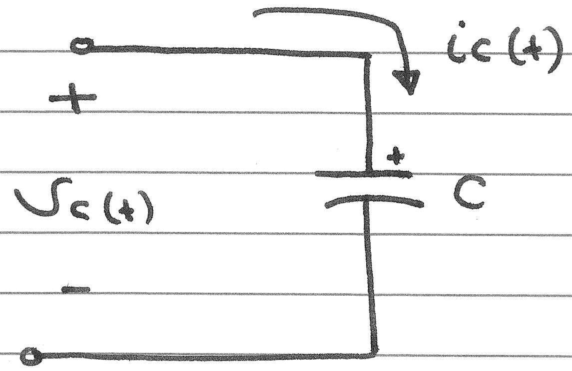
Chapter 7

"Response of First_Order RL and RC Circuits"

Capacitors and Inductors

- Resistors are passive elements which dissipate energy only.
- Two importan passive Linear circuit elements : Capacitor , and Inductor
- Capacitors and Inductors do not dissipate but store energy , which can be retrieved at a Later time
- Capacitors and Inductors are called Storage elements.
- $W_c(t) = \frac{1}{2} C V_c^2(t)$
- $W_L(t) = \frac{1}{2} L i_L^2(t)$

Some of the important characteristics of a Capacitor



$$i_c(t) = C \frac{dv_c(t)}{dt}$$

$$v_c(t) = v_c(0^-) + \frac{1}{C}\int_0^t i_c(t)\,dt \quad , \text{ for } t \geq 0$$

1. The current through a capacitor is zero if the voltage across it is not changing with time. A Capacitor is therefore an open Circuit to dc.

2. A finite amount of energy can be stored in a Capacitor even if the current through the Capacitor is zero.

3. The Capacitor never dissipate energy, but only store it.

$$v_c(t) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i_c(t)\,dt \,, \quad \text{for } t \geq 0$$

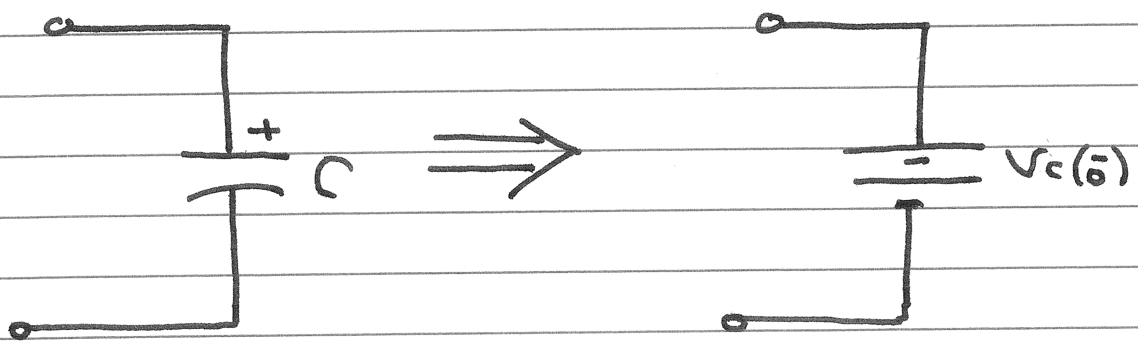
at $t = 0^+$

$$v_c(0^+) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{0^+} i_c(t)\,dt$$

$$v_c(0^+) = v_c(0^-)$$

4. It is impossible to change the voltage across a capacitor by a finite amount in zero time, for this requires an infinite current through the capacitor

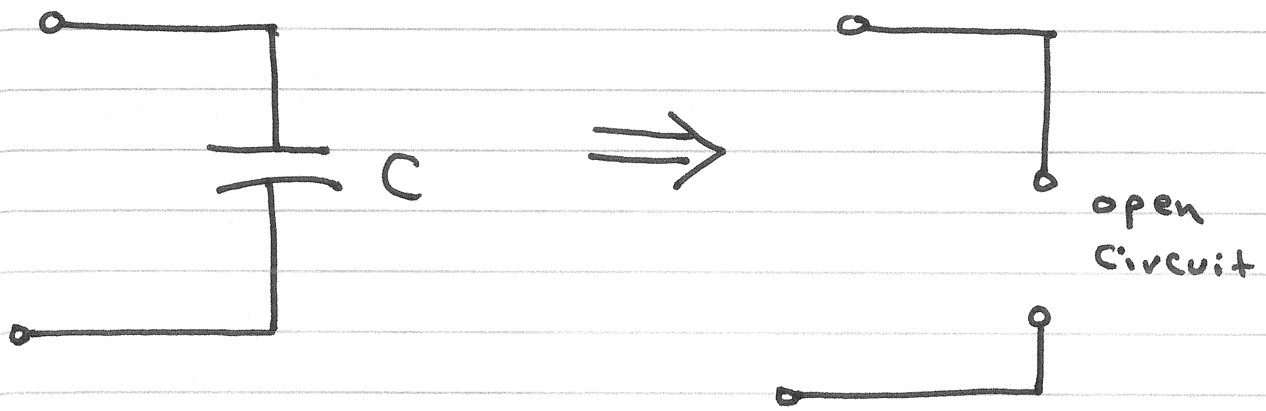
5. At $= 0^+$



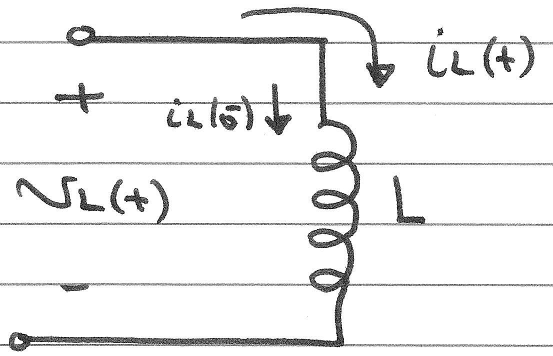
$$i_C(t) = C\,\frac{dv_C(t)}{dt}$$

A capacitor is therefore an open circuit to dc

At $t = 0^-$, and $t = \infty$ (After the change)



Some of the important characteristics of an inductor



$$v_L(t) = L \frac{di_L(t)}{dt}$$

$$i_L(t) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v_L(t)\,dt \quad ; \text{for } t \geq 0$$

1. There is no voltage across an inductor if the current through it is not changing with time.

   An inductor is therefore a short circuit to dc

2. A finite amount of energy can be stored in an inductor even if the voltage across the inductor is zero.

3. The inductor never dissipate energy, but only

Store it

4. $$i_L(t) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{t} v_L(t)\,dt \quad ; \text{ for } t \geq 0$$

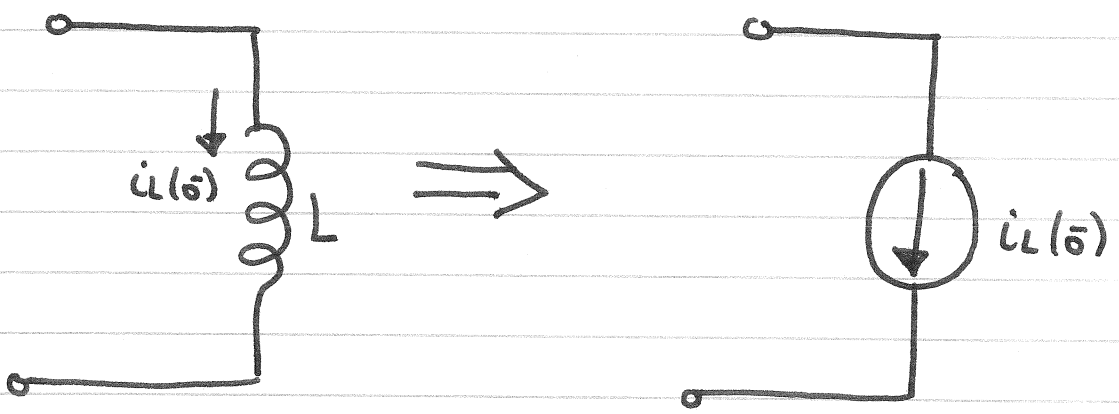
at $t = 0^+$

$$i_L(0^+) = i_L(0^-) + \frac{1}{L}\int_{0^-}^{0^+} v_L(t)\,dt$$

$$\therefore \quad i_L(0^+) = i_L(0^-)$$

It is impossible to change the current through an inductor by a finite amount in zero time, for this requires an infinite voltage across the inductor.

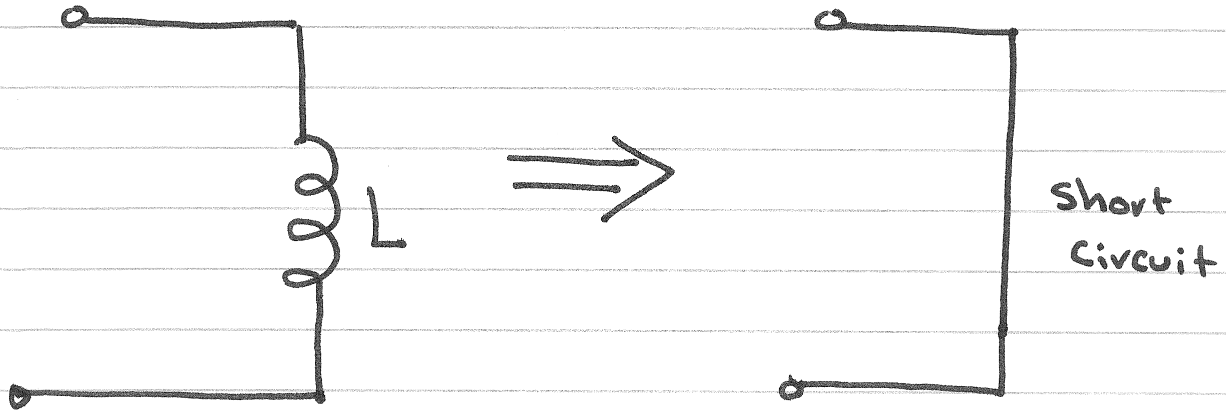
At $t = 0^+$



$$v_L(t) = L\,\frac{di_L(t)}{dt}$$

An inductor is therefore a short circuit to dc

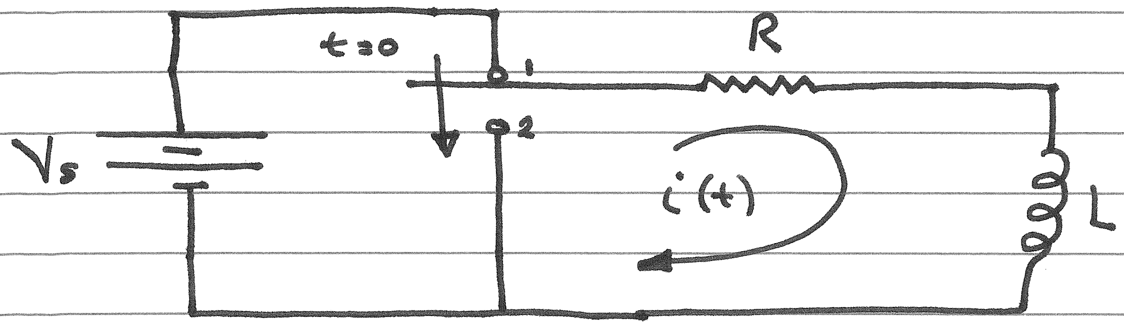
At $t = 0^-$, and $t = \infty$ (After the change)



First order Circuit

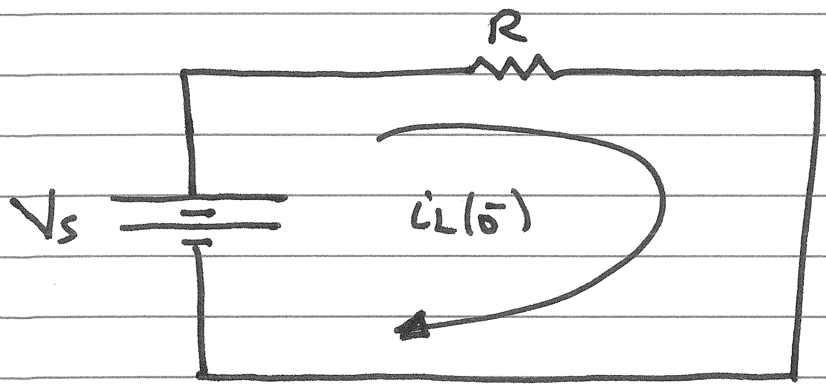
- A first-order circuit can only contain one energy storage element (a capacitor or an inductor) or a combination of capacitors or inductors that can be reduced to one capacitor or inductor
- The circuit will also contain one or more resistances
- A first-order circuit is characterized by a first-order differential equation

Natural Response of First-Order Circuit



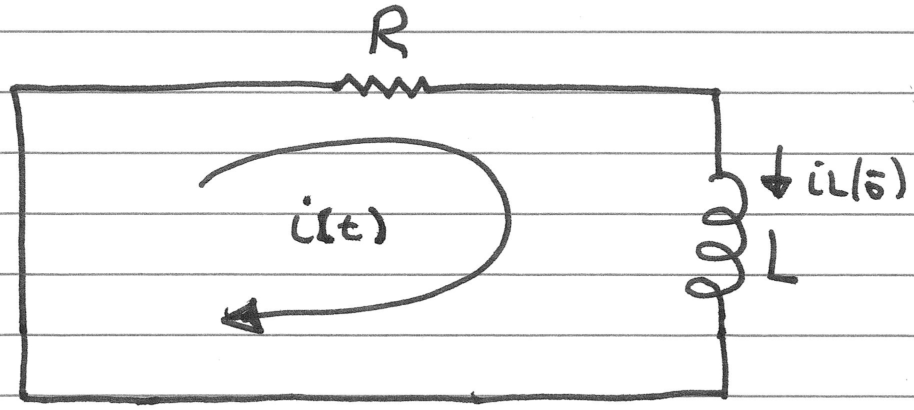
Find $i(t)$ for $t > 0$

1) For $t < 0$; $t = \bar{0}$

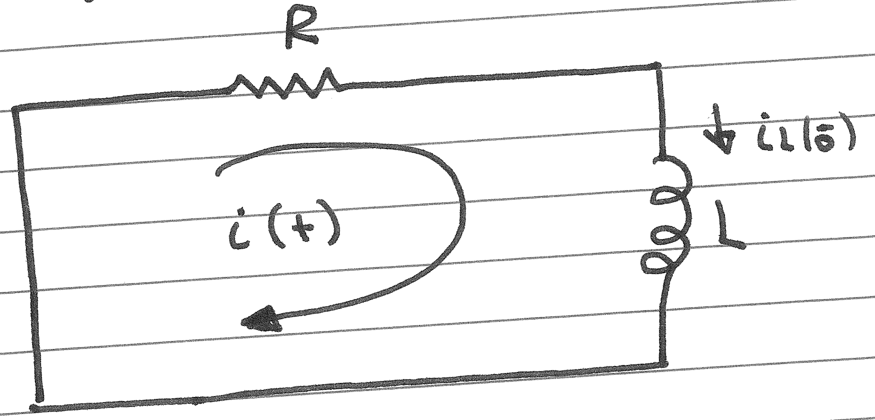


$$i_L(\bar{0}) = \frac{V_s}{R}$$

2) For $t > 0$



For $t > 0$



KVL :

$$R\,i(t) + L\frac{di(t)}{dt} = 0$$

homogeneous first order differential equation

$$i(t) = A e^{st} \quad \text{for } t > 0$$

$$RAe^{st} + LASe^{st} = 0$$

$$Ae^{st}(R + LS) = 0$$

$$\therefore S = -\frac{R}{L}$$

To find A :

$$i(t) = A e^{st} \qquad t > 0$$

$$i(0^+) = A$$

$$i(0^+) = i_L(0^+) = i_L(0^-)$$

$$\therefore A = i_L(0^-) = \frac{V_s}{R}$$

-11-

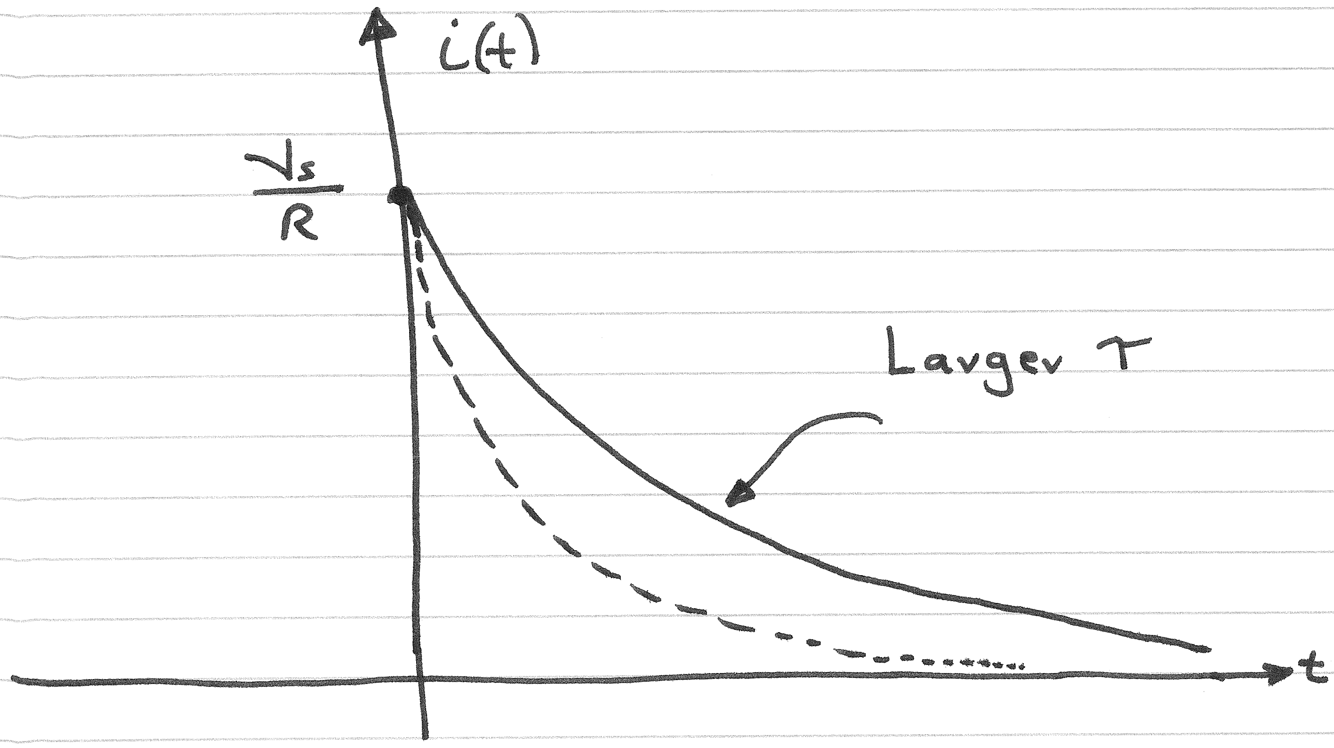
$$\therefore \quad i(t) = \frac{V_s}{R} e^{-\frac{R}{L}t} \qquad t > 0$$

let $\tau = \frac{L}{R}$

$\tau \equiv$ Time Constant

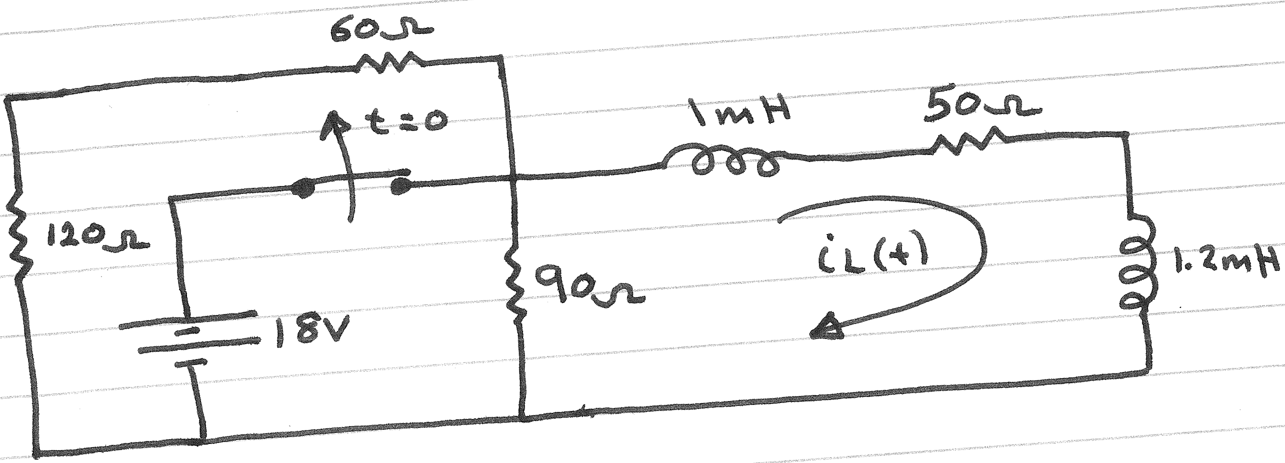
$$\therefore \quad i(t) = \frac{V_s}{R} e^{-\frac{t}{\tau}} \qquad t > 0$$

t	$i(t)$
0	$\frac{V_s}{R}$
τ	$0.37 \frac{V_s}{R}$
2τ	$0.14 \frac{V_s}{R}$
3τ	$0.05 \frac{V_s}{R}$
4τ	$0.018 \frac{V_s}{R}$
5τ	$0.0067 \frac{V_s}{R}$

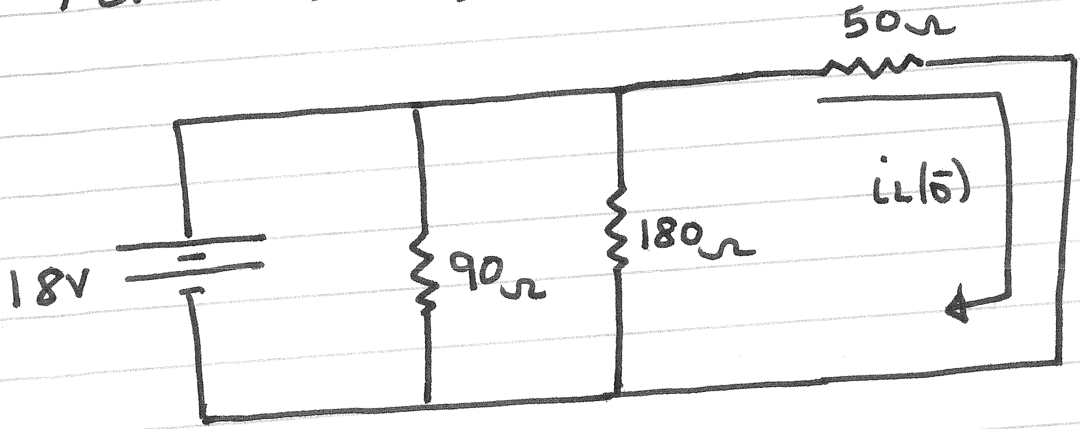


i(t)
Vs/R
Larger T
t

Find $i_L(t)$ for $t > 0$

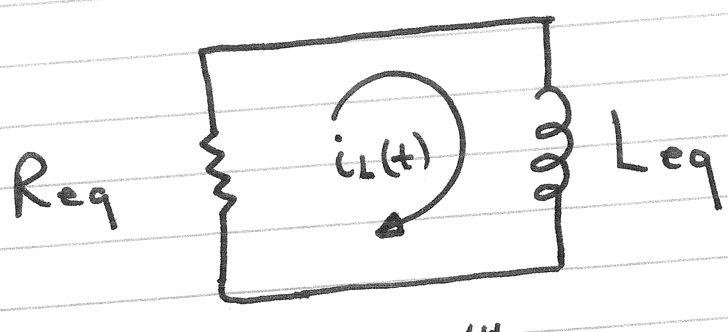


1) For $t < 0$; $t = 0^-$



$$i_L(0^-) = \frac{18V}{50\Omega} = 0.36 A$$

2) For $t > 0$



$$R_{eq} = 90\Omega \parallel 180\Omega + 50\Omega = 110\Omega$$

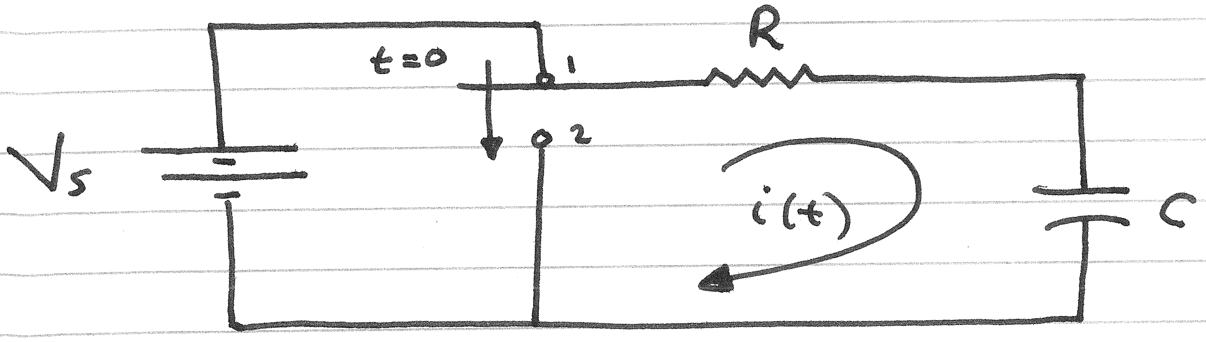
$$L_{eq} = 1mH + 1.2mH = 2.2mH$$

$$\tau = \frac{L_{eq}}{R_{eq}} = 20\mu s$$

$$\therefore i_L(t) = A e^{-t/\tau} \quad \text{for } t > 0$$

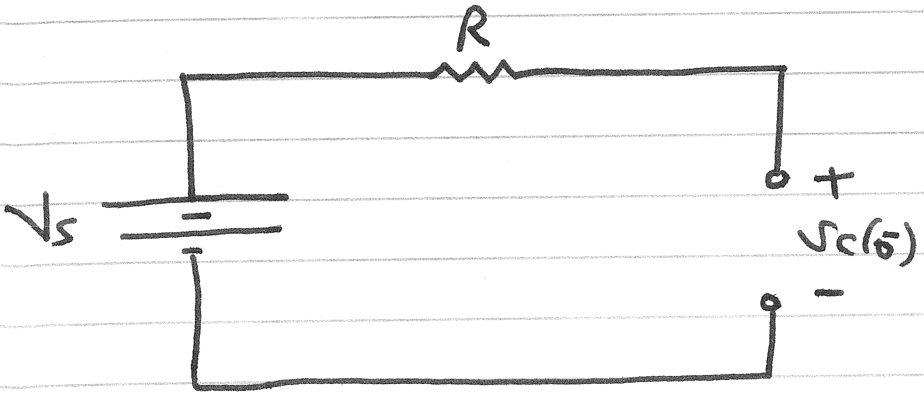
$$i_L(t) = 0.36 e^{-50{,}000t} \quad \text{for } t > 0$$

RC Circuit



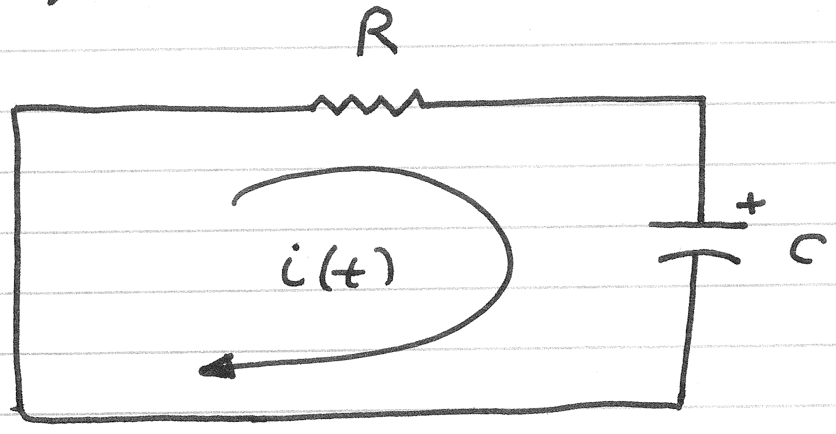
Find $i(t)$ for $t > 0$

1) For $t < 0$; $t = 0^-$

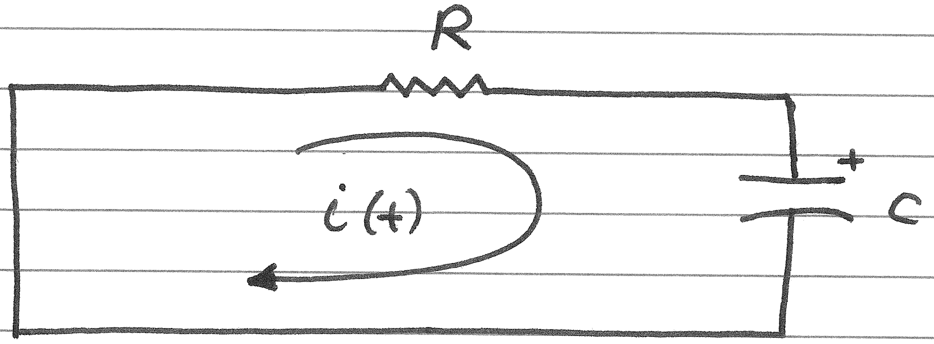


$v_c(0^-) = V_s$

2) For $t > 0$



For $t > 0$



KVL :

$$R\,i(t) + V_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt \qquad t > 0$$

$$R\frac{di(t)}{dt} + \frac{1}{C}\,i(t) = 0$$

homogeneous First order differential equation

$$\therefore \quad i(t) = A e^{st} \qquad t > 0$$

$$RAse^{st} + \frac{1}{C}Ae^{st} = 0$$

$$Ae^{st}\left(Rs + \frac{1}{C}\right) = 0$$

$$\therefore \quad s = -\frac{1}{RC}$$

$$\therefore \quad i(t) = A e^{-\frac{1}{RC}t} \qquad t > 0$$

let $\tau = RC$ = time constant

$$\therefore \quad i(t) = A e^{-t/\tau} \qquad t > 0$$

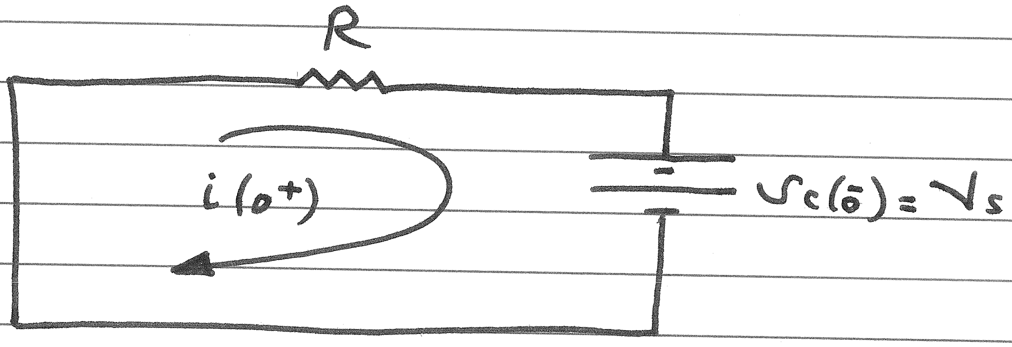
To find A

$$i(t) = A e^{-t/\tau} \qquad t > 0$$

$$i(0^+) = A$$

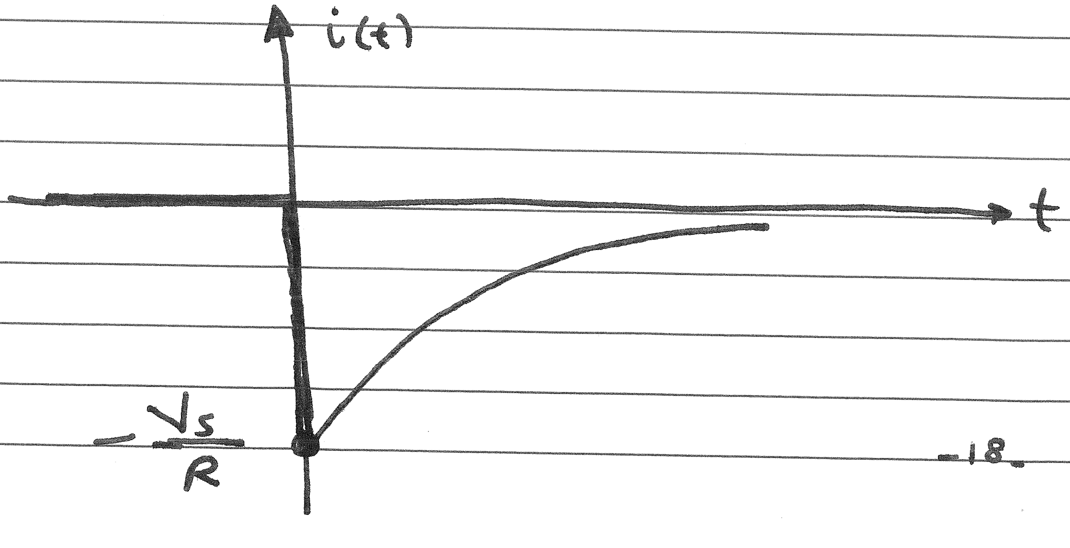
To find $i(0^+)$

at $t = 0^+$

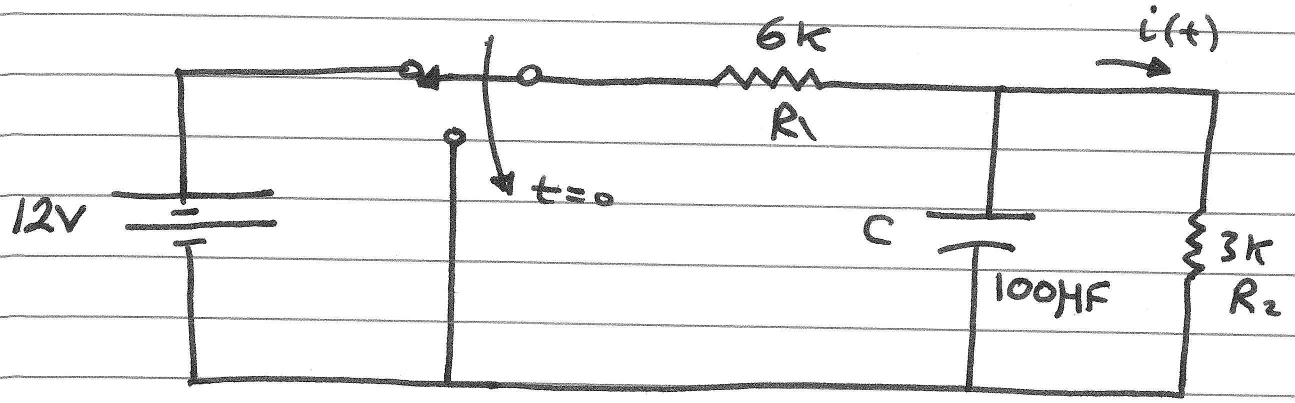


$$i(0^+) = -\frac{v_c(0^-)}{R} = -\frac{V_s}{R}$$

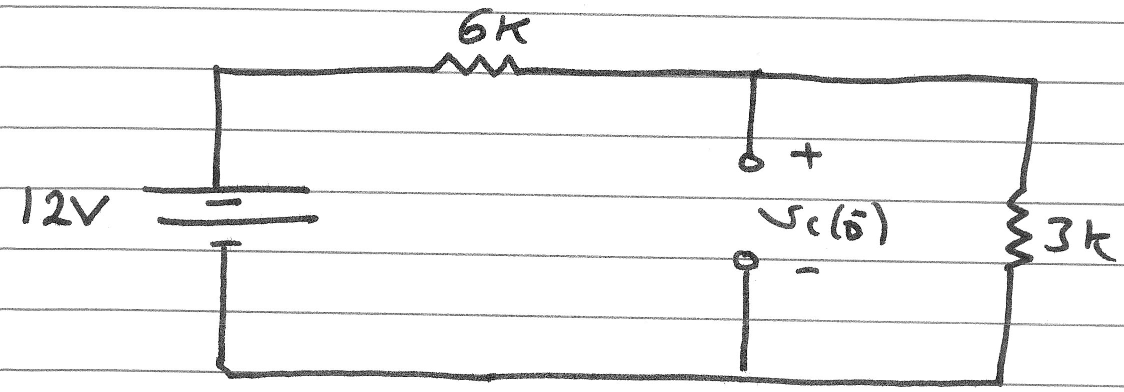
$$\therefore\ i(t) = -\frac{V}{R} e^{-t/\tau}$$



Calculate $v_c(t)$ and $i(t)$ for $t > 0$

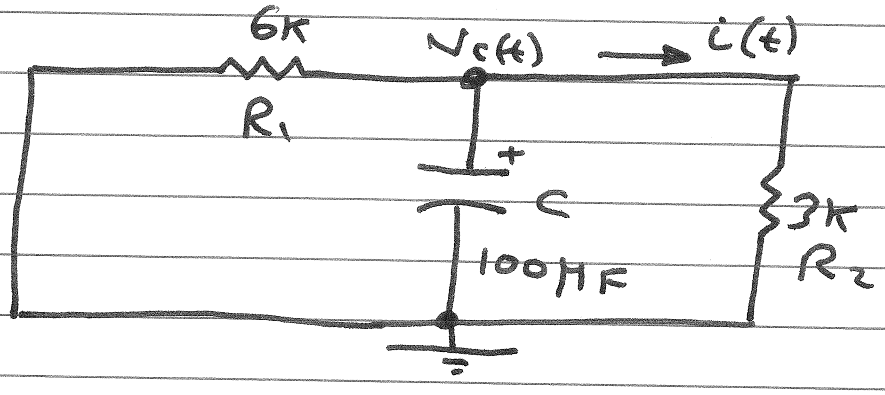


1) For $t < 0$; $t = 0^-$



$$v_c(0^-) = \frac{3k}{3k + 6k} \, 12V = 4V$$

2) For $t > 0$



KCL :

$$\frac{v_c(t)}{6k} + \frac{v_c(t)}{3k} + C\frac{dv_c(t)}{dt} = 0$$

$$\frac{dv_c(t)}{dt} + 5v_c(t) = 0$$

$$\therefore \; v_c(t) = A e^{-t/\tau} \qquad t > 0$$

$$\tau = R_{eq} C$$

$$R_{eq} = 6k \| 3k = 2k$$

$$C = 100 \mu F$$

$$\tau = R_{eq} C = 0.2 \text{ sec}$$

$$v_c(t) = A e^{-t/0.2} \qquad t > 0$$

<u>To find A</u>

$$v_c(0^+) = A = v_c(0^-) = 4V$$

$$\therefore \; v_c(t) = 4 e^{-5t} \qquad t > 0$$

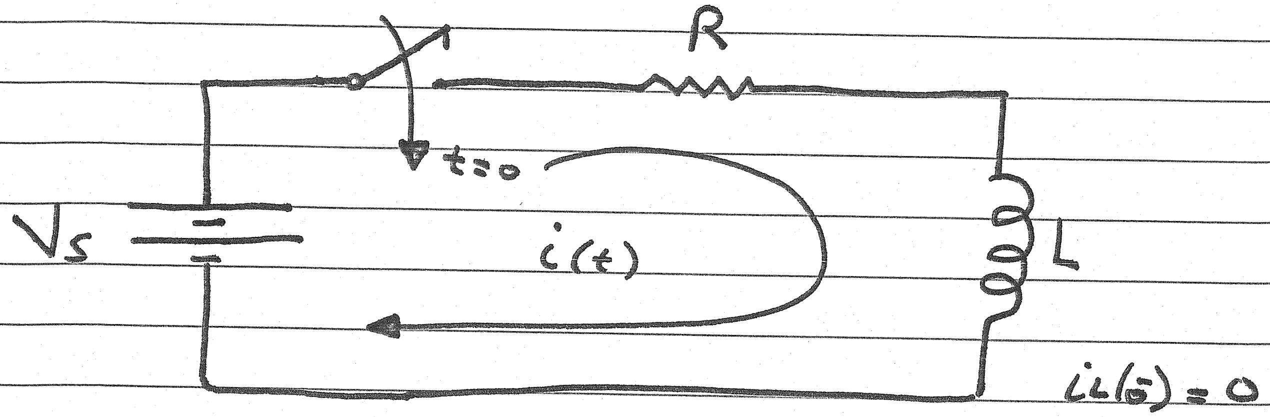
$$i(t) = \frac{v_c(t)}{R_2}$$

$$i(t) = \frac{4}{3} e^{-5t} \text{ mA} \qquad t > 0$$

The step Response of RC and RL Circuits

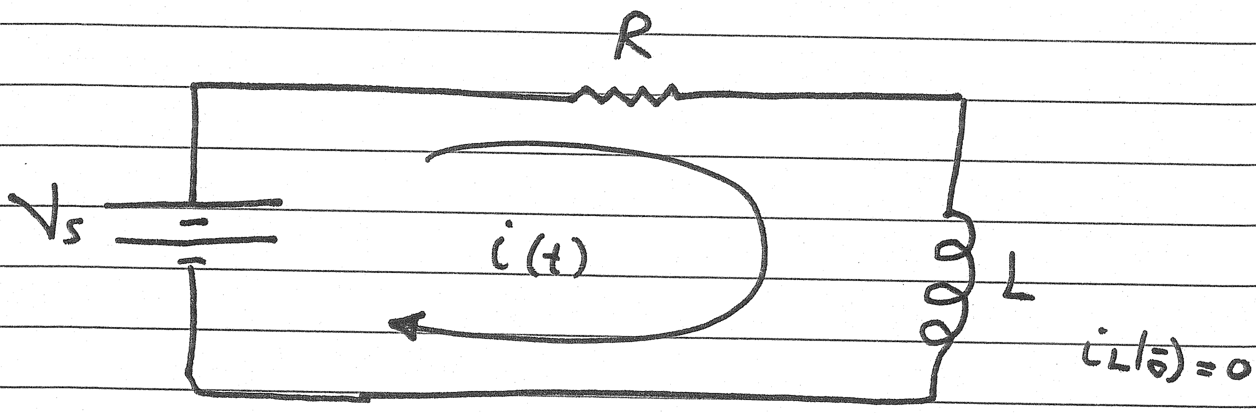
The response of a Circuit to the sudden application of a Constant voltage or Current source is referred to as the Step response of the Circuit.

The step response of an RL Circuit



Find $i(t)$ for $t > 0$

For $t > 0$



KVL :

$$V_s = R\,i(t) + L\frac{di(t)}{dt} \qquad t > 0$$

non homogenous First order differential equation

$$\therefore \quad i(t) = i_n(t) + i_f(t)$$

$i_n(t) \equiv$ natural response

$i_f(t) \equiv$ forced response

<u>To find $i_f(t)$</u>

let $i_f(t) = K$

$$V_s = R\,i(t) + L\,\frac{d\,i(t)}{dt}$$

$$V_s = RK + L(0)$$

$$V_s = RK$$

$$\therefore K = \frac{V_s}{R} = i_f(t)$$

now

$$i(t) = i_n(t) + i_f(t) \qquad t > 0$$

$$i(t) = A e^{-t/\tau} + K \qquad t > 0$$

$$\tau = \frac{L}{R}$$

$$i(t) = A e^{-t/\tau} + \frac{V_s}{R} \qquad t > 0$$

To find A

$$i(t) = \frac{V_s}{R} + A e^{-t/\tau} \qquad t > 0$$

$$i(0^+) = \frac{V_s}{R} + A$$

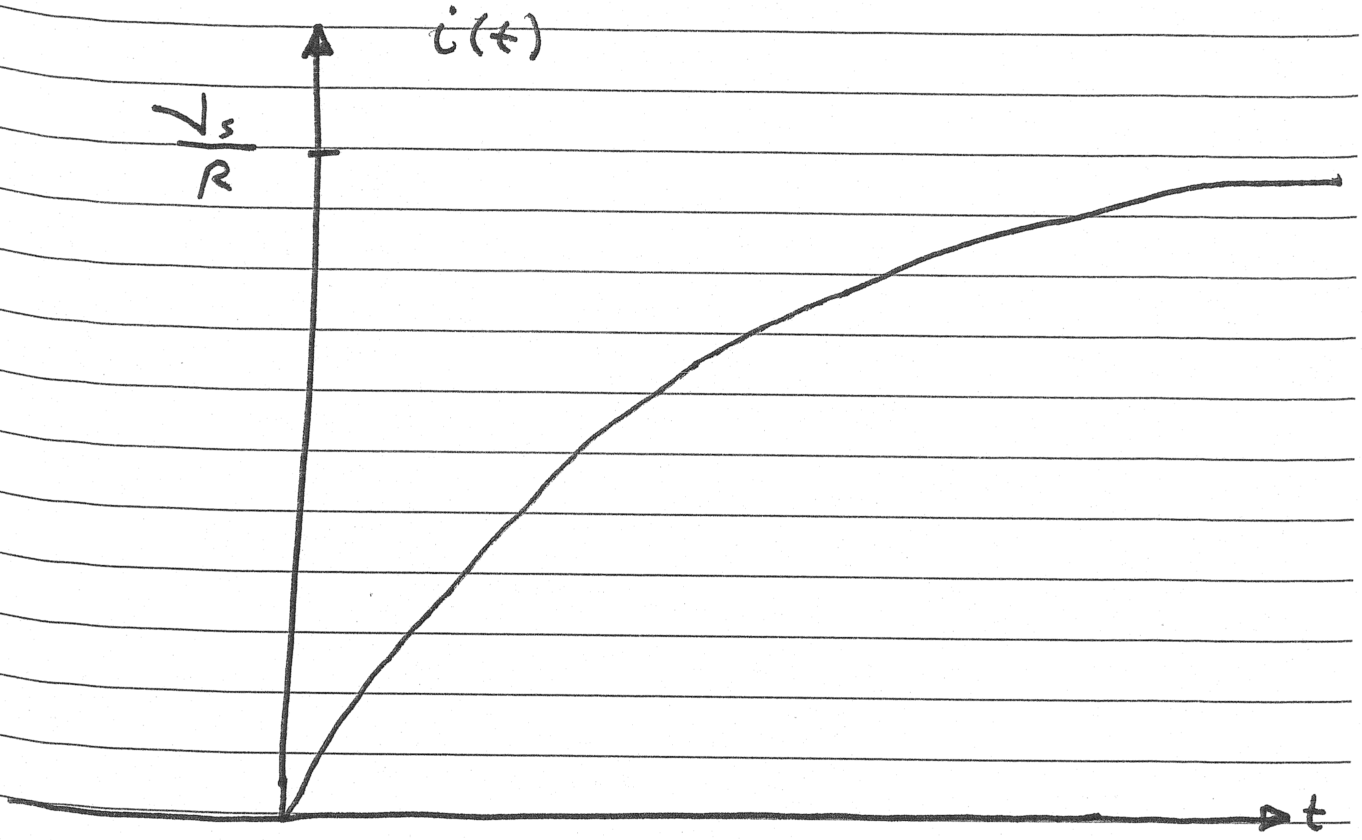
But $i(0^+) = i_L(0^+) = i_L(0^-) = 0$

$$\therefore \quad A = -\frac{V_s}{R}$$

$$\therefore \quad i(t) = \frac{V_s}{R} - \frac{V_s}{R} e^{-t/\tau} \qquad t \geq 0$$

$$i(t) = \frac{V_s}{R}\left(1 - e^{-t/\tau}\right) \qquad t \geq 0$$

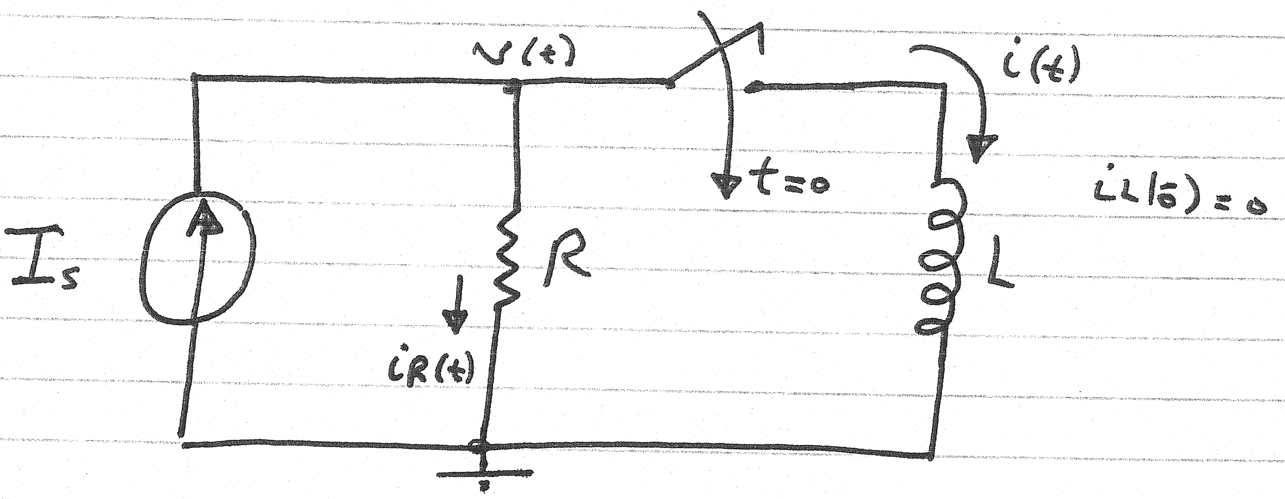
t	$i(t)$
0	0
τ	$0.63 \frac{V_s}{R}$
2τ	$0.86 \frac{V_s}{R}$
3τ	$0.95 \frac{V_s}{R}$
4τ	$0.98 \frac{V_s}{R}$
5τ	$0.99326 \frac{V_s}{R}$



$$i(t) = \frac{V_s}{R}\left(1 - e^{-t/\tau}\right)$$

$$\tau = \frac{L}{R}$$

Find $i(t)$ for $t > 0$



For $t > 0$

KCL :

$$I_s = i_R(t) + i(t) \qquad t > 0$$

$$I_s = \frac{v(t)}{R} + i(t) \qquad t > 0$$

$$v(t) = v_R(t) = v_L(t) = L\frac{di(t)}{dt}$$

$$I_s = \frac{L}{R}\frac{di(t)}{dt} + i(t) \qquad t > 0$$

$$\therefore\ i(t) = i_n(t) + i_f(t)$$

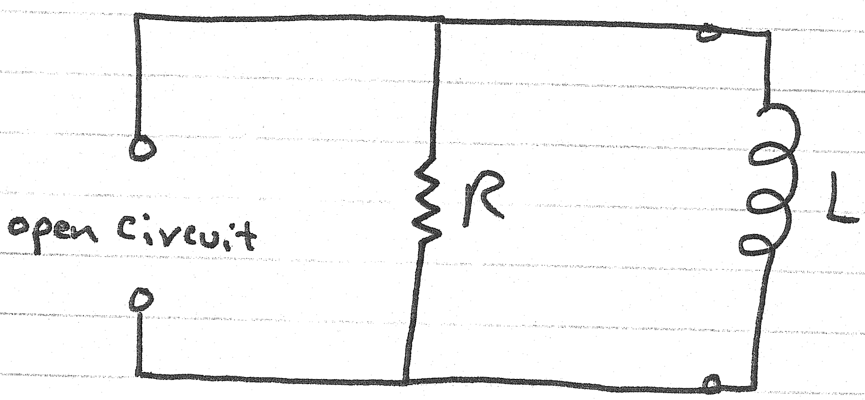
Let $i_f(t) = K$

$$I_s = \frac{L}{R}\cdot 0 + K$$

$$\therefore\ K = I_s = i_f(t)$$

$$\tau = \frac{L}{R_{eq}}$$

R_{eq} = The Thevenin resistance seen by the inductor



$$\therefore R_{eq} = R$$

$$i(t) = K + A e^{-t/\tau} \quad ; \; t > 0$$

$$i(t) = I_s + A e^{-t/\tau} \quad ; \; t > 0$$

<u>To find A</u>

$$i(0^+) = i_L(0^+) = i_L(0^-) = 0$$

$$i(0^+) = I_s + A = i_L(0^-) = 0$$

$$\therefore A = -I_s$$

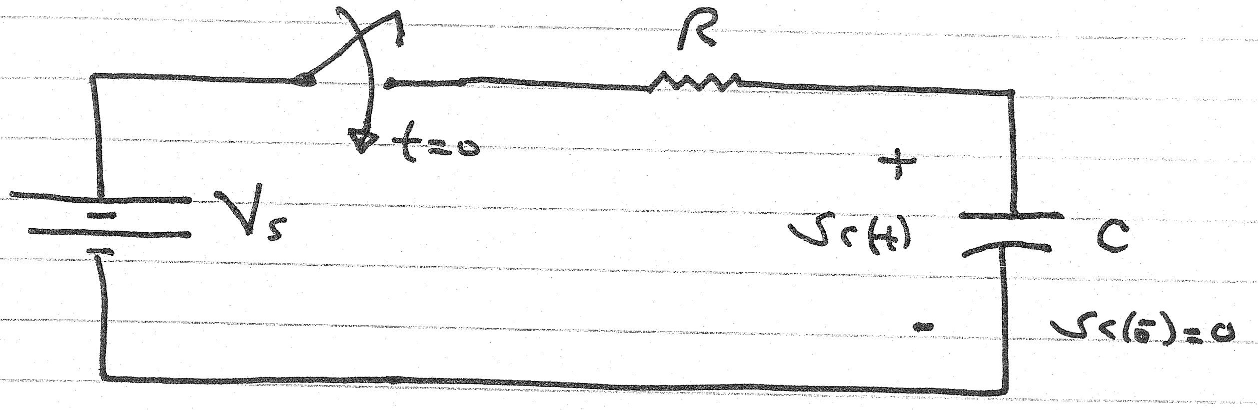
$$\therefore\ i(t) = i_n(t) + i_f(t) \qquad ; t > 0$$

$$i(t) = A e^{-t/\tau} + I_s \qquad ; t > 0$$

$$i(t) = -I_s e^{-t/\tau} + I_s \qquad ; t > 0$$

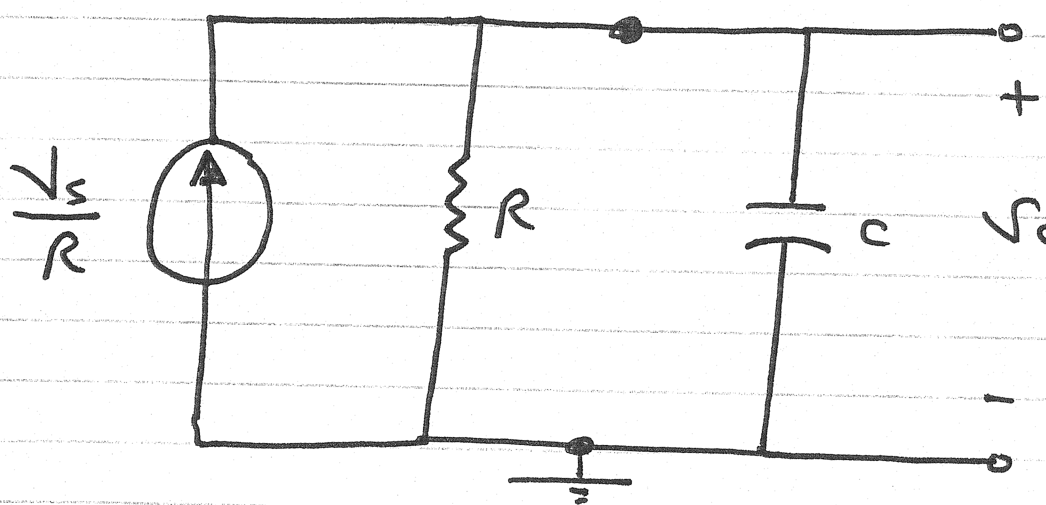
$$\therefore\ i(t) = I_s \left(1 - e^{-t/\tau} \right) \qquad t \geq 0$$

The step response of an RC Circuit



Find $v_c(t)$ for $t > 0$

For $t > 0$



KCL :

$$\frac{V_s}{R} = i_R(t) + i_c(t) \qquad t > 0$$

$$\frac{V_s}{R} = \frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt} \qquad t > 0$$

$$\frac{V_s}{R} = \frac{v_c(t)}{R} + C\frac{dv_c(t)}{dt}$$

First order nonhomogenous differential equation

$$\therefore \quad v_c(t) = v_{cn}(t) + v_{cf}(t) \qquad t > 0$$

$$v_c(t) = A e^{-t/\tau} + K$$

<u>To find K</u>

$$\frac{V_s}{R} = \frac{K}{R} + 0$$

$$\therefore K = V_s$$

$$\therefore \quad v_c(t) = A e^{-t/\tau} + V_s \qquad t > 0$$

$$\tau = R_{eq} C$$

$$R_{eq} = R$$

<u>To find A</u>

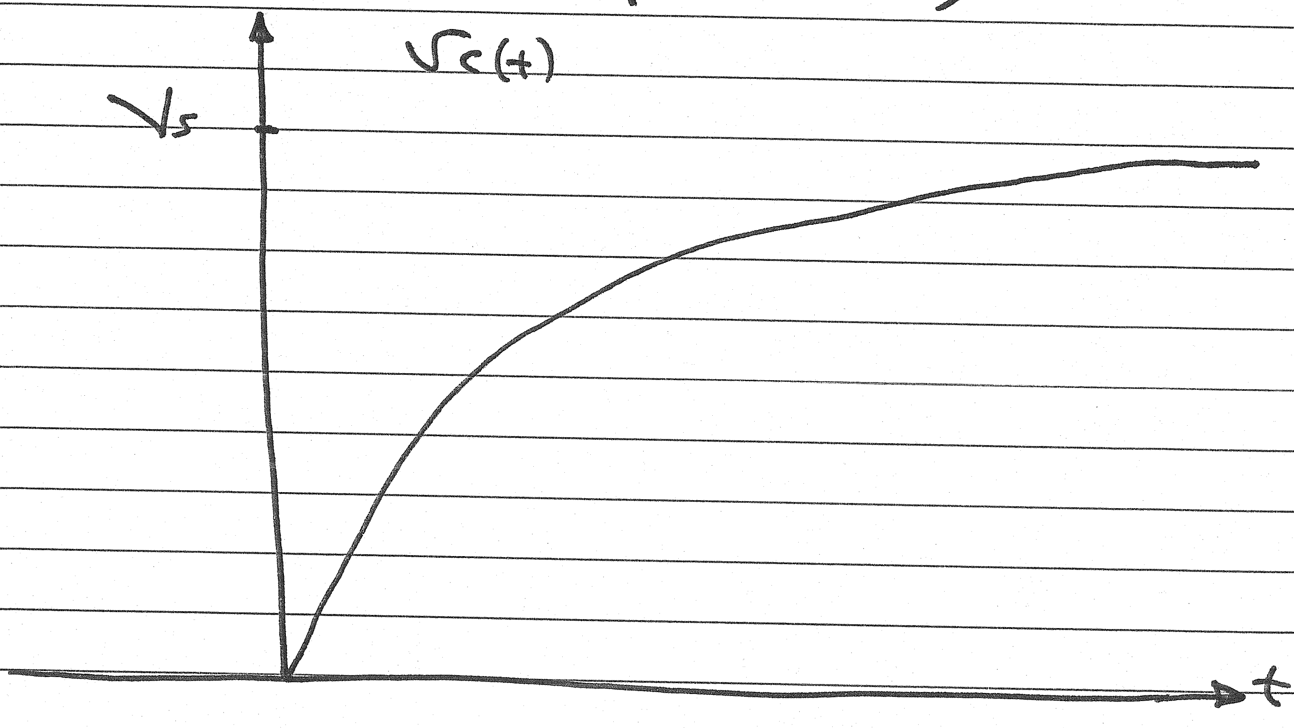
$$v_c(0^+) = v_c(0^-) = 0$$

$$V_c(0^+) = K + A = V_c(0^-) = 0$$

$$\therefore A = -K = -V_s$$

$$\therefore V_c(t) = V_s - V_s e^{-t/\tau} \qquad t \geq 0$$

$$V_c(t) = V_s \left(1 - e^{-t/\tau}\right) \qquad t \geq 0$$



When all independent sources are Constant, the response of the First order Circuit has the form

$$r(t) = r_n(t) + K \qquad t>0$$

$$r(t) = A e^{-t/\tau} + K \qquad t>0$$

$$r(\infty) = K$$

$$\therefore \ r(t) = A e^{-t/\tau} + r(\infty) \qquad t>0$$

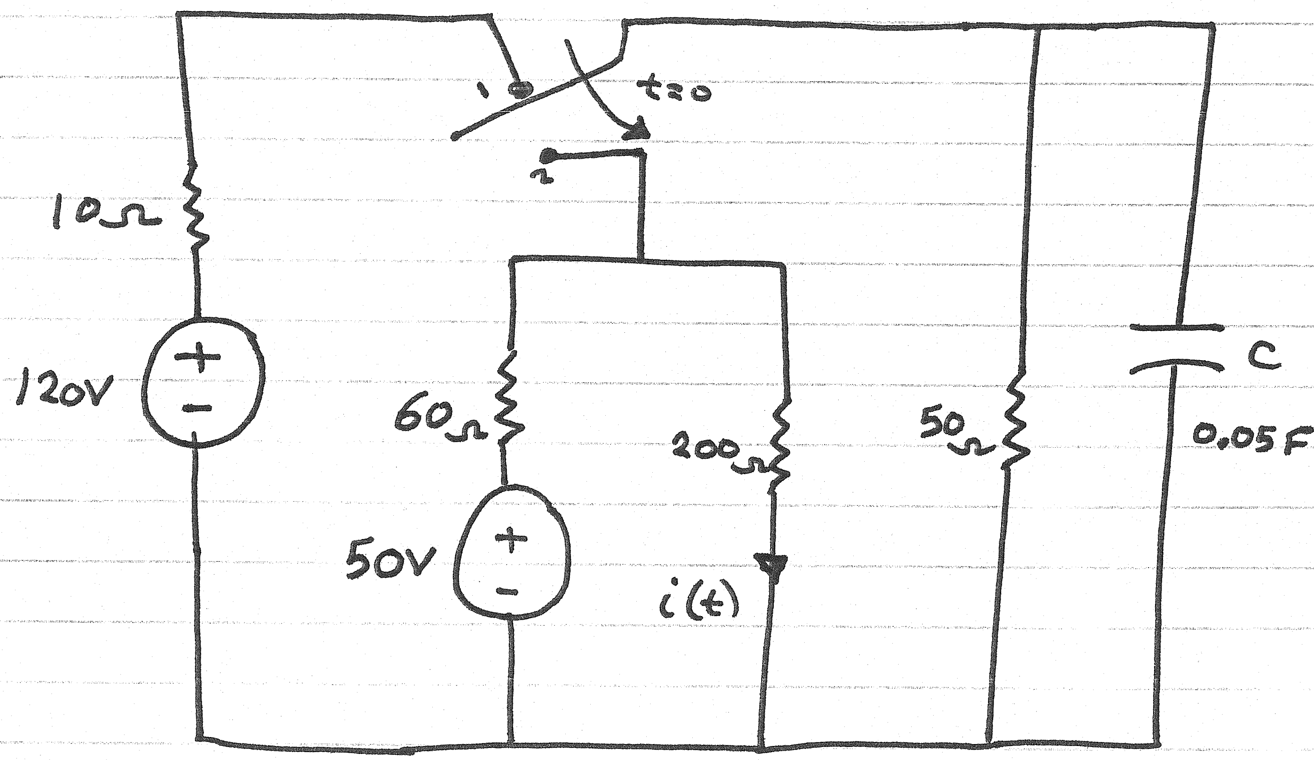
$$r(0^+) = A + r(\infty)$$

$$\therefore \ A = r(0^+) - r(\infty)$$

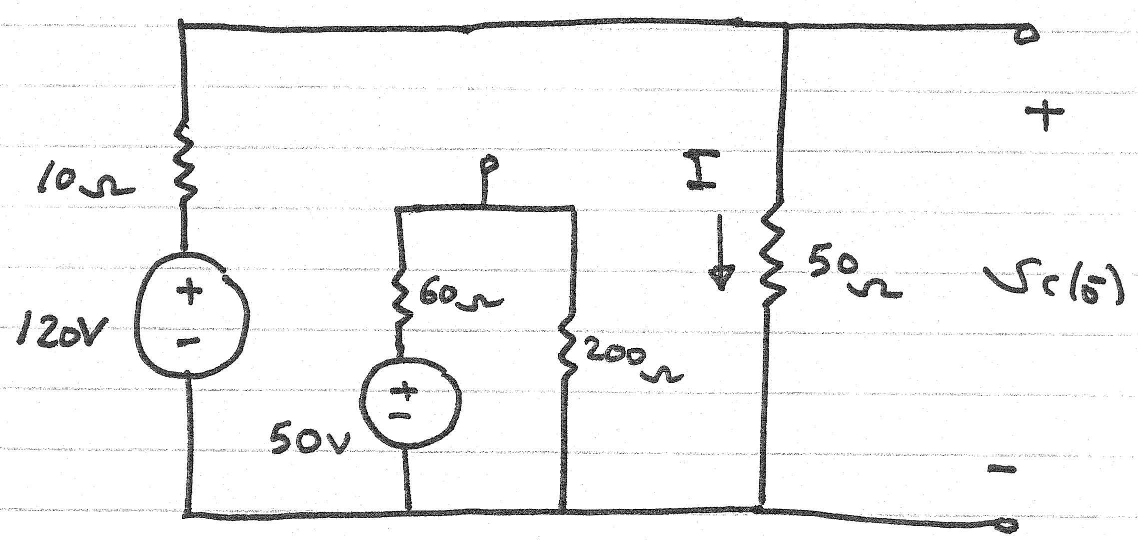
$$\therefore \ r(t) = r(\infty) + \left[r(0^+) - r(\infty)\right] e^{-t/\tau} \qquad t>0$$

$$\tau = \frac{L}{R_{eq}} \quad \text{or} \quad \tau = R_{eq}\, C$$

Find $i(t)$ for $t > 0$



For $t < 0$; $t = 0^-$

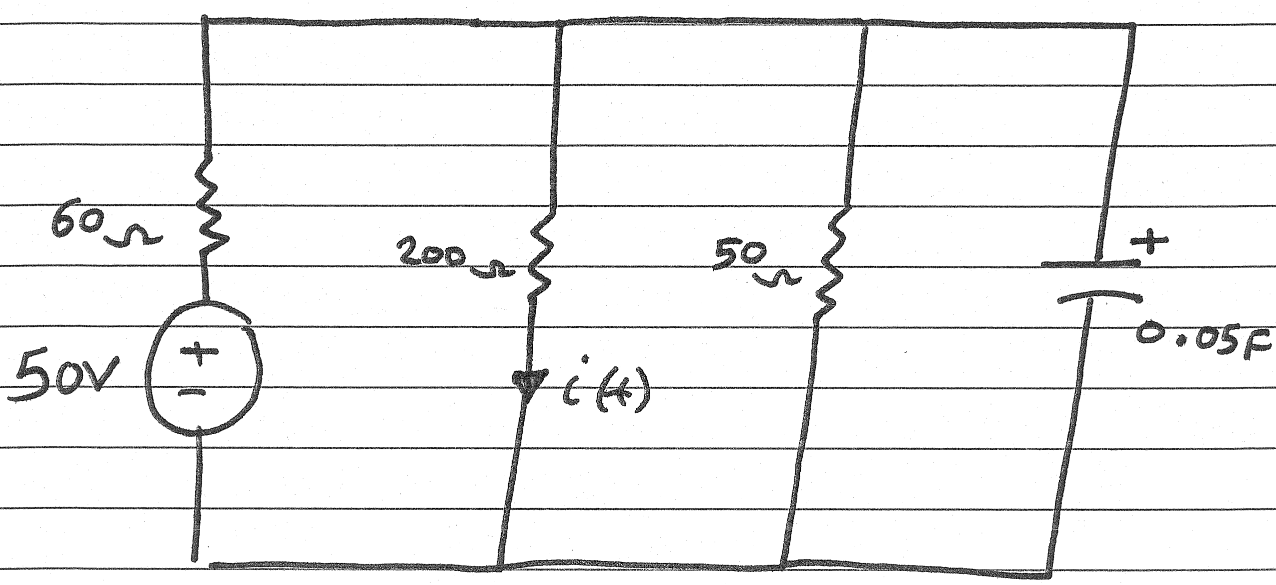


KVL : $-120 + 10I + 50I = 0$

$\therefore\ I = 2A$

$V_c(0^-) = 50 I = 100V$

For $t > 0$

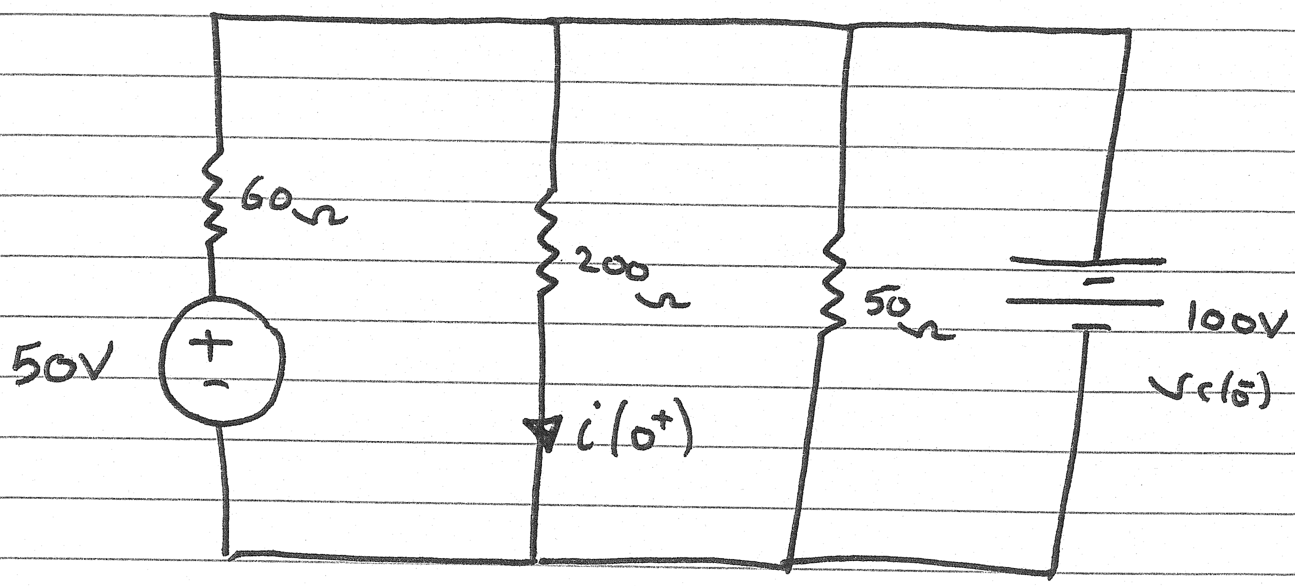


$$r(t) = r(\infty) + [r(0^+) - r(\infty)]\, e^{-t/\tau} \quad t > 0$$

$$i(t) = i(\infty) + [i(0^+) - i(\infty)]\, e^{-t/\tau} \quad t > 0$$

To find $i(0^+)$

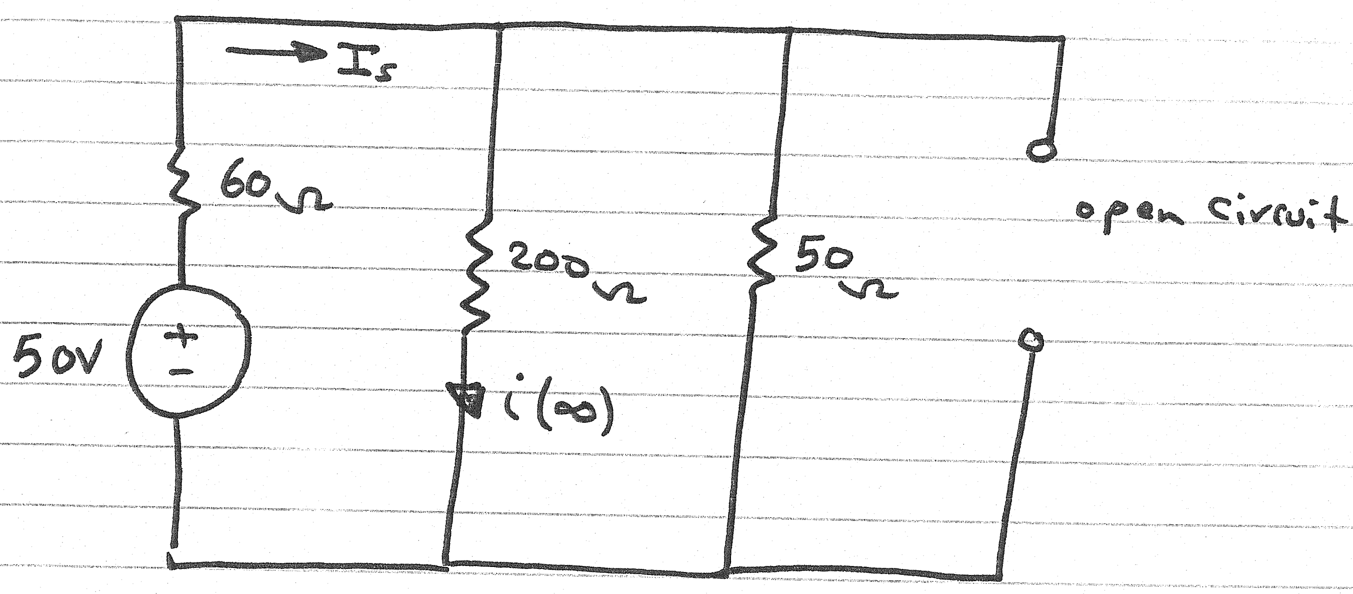
at $t = 0^+$



$$i(0^+) = \frac{v_c(0^-)}{200} = \frac{100}{200} = 0.5\,A$$

To find $i(\infty)$

at $t = \infty$



$$i(\infty) = \frac{50}{50+200} I_s$$

$$I_s = \frac{50}{50\Omega \parallel 200\Omega + 60\Omega} = 0.5A$$

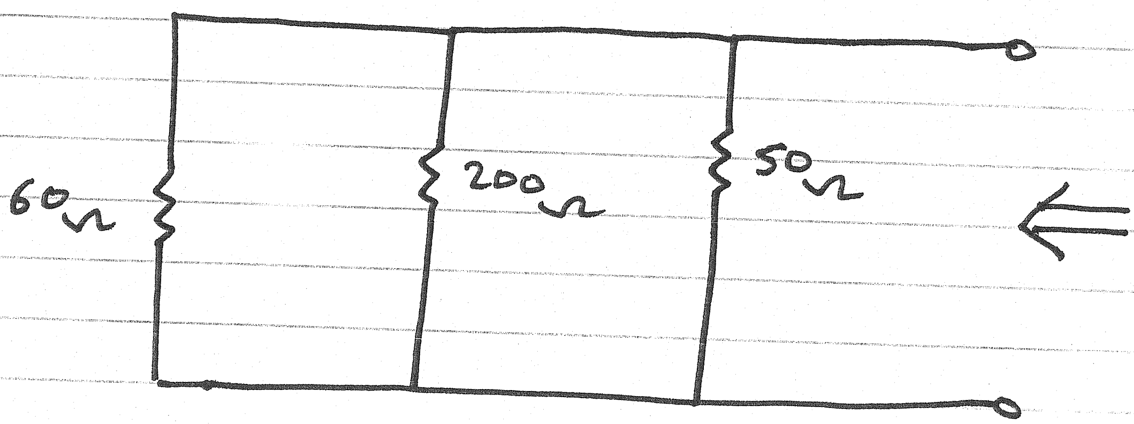
$\therefore \; i(\infty) = 0.1A$

To find τ

for $t > 0$

$$\tau = R_{eq} C$$

$R_{eq} = R_{TH}$ seen by the Capacitor



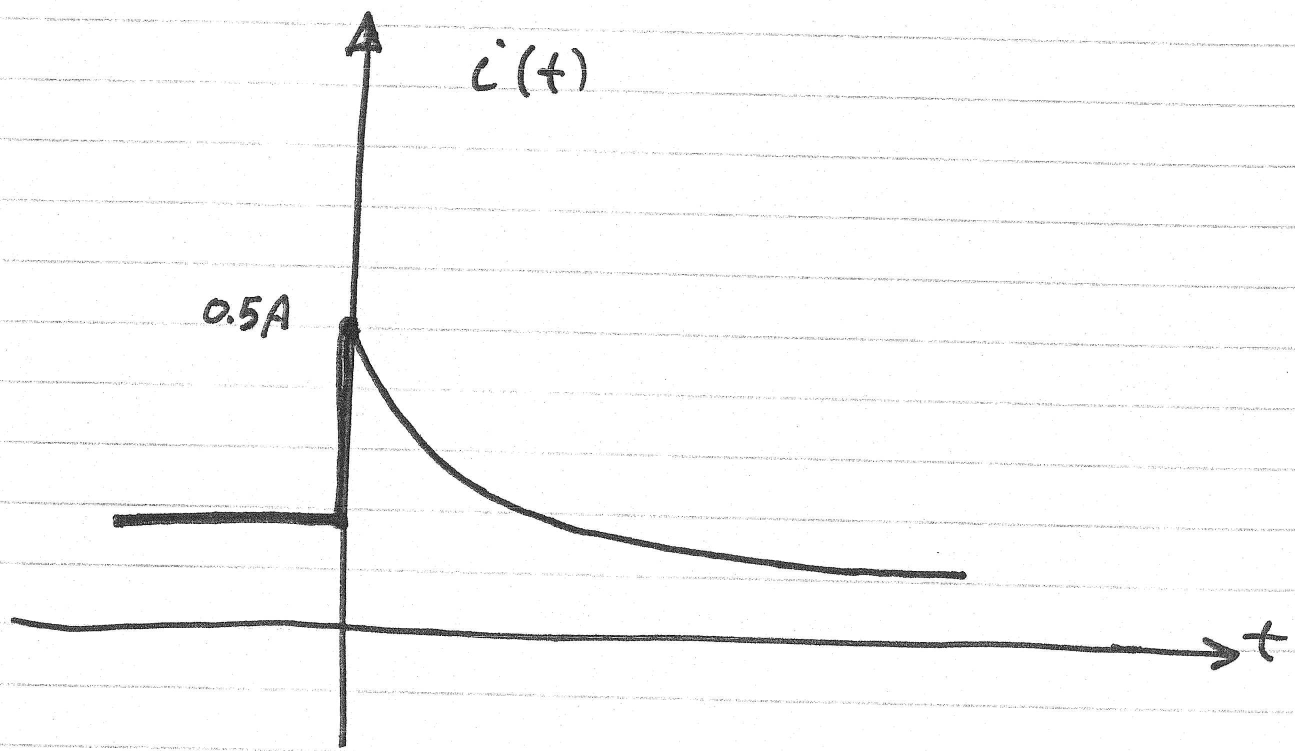
$$R_{TH} = 60\,\Omega \parallel 200\,\Omega \parallel 50\,\Omega$$

$$R_{TH} = 24\,\Omega$$

$$\tau = (0.05)(24) = 1.2 \text{ sec}$$

$$\therefore\ i(t) = \left(0.1 + 0.4\, e^{-t/1.2}\right) A \ ; t > 0$$

$$i(t) = \left(0.1 + 0.4\, e^{-t/1.2}\right) \; A \qquad ; t > 0$$



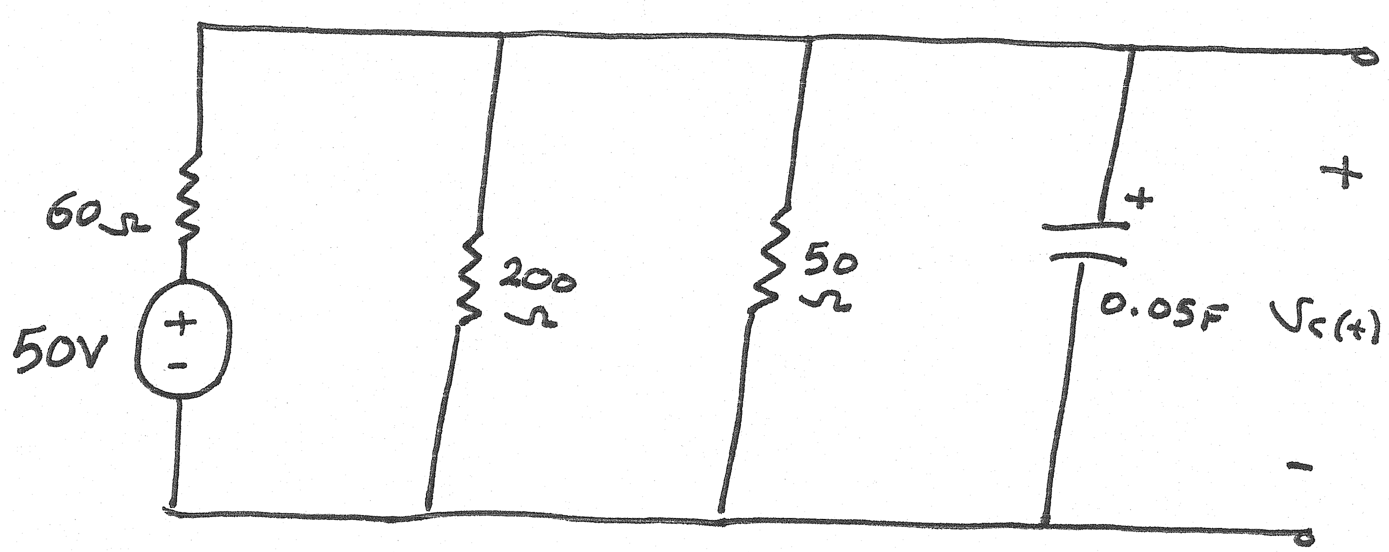
$i(0^-) = 0.192\,A$

$i(0^+) = 0.5\,A$

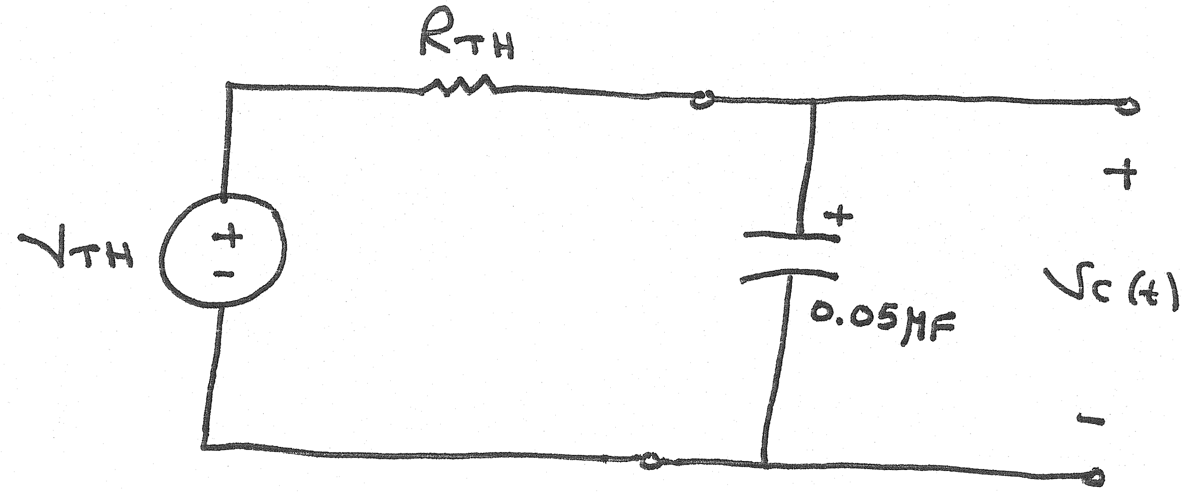
$i(\infty) = 0.1\,A$

To find $v_c(t)$ for $t > 0$

For $t > 0$

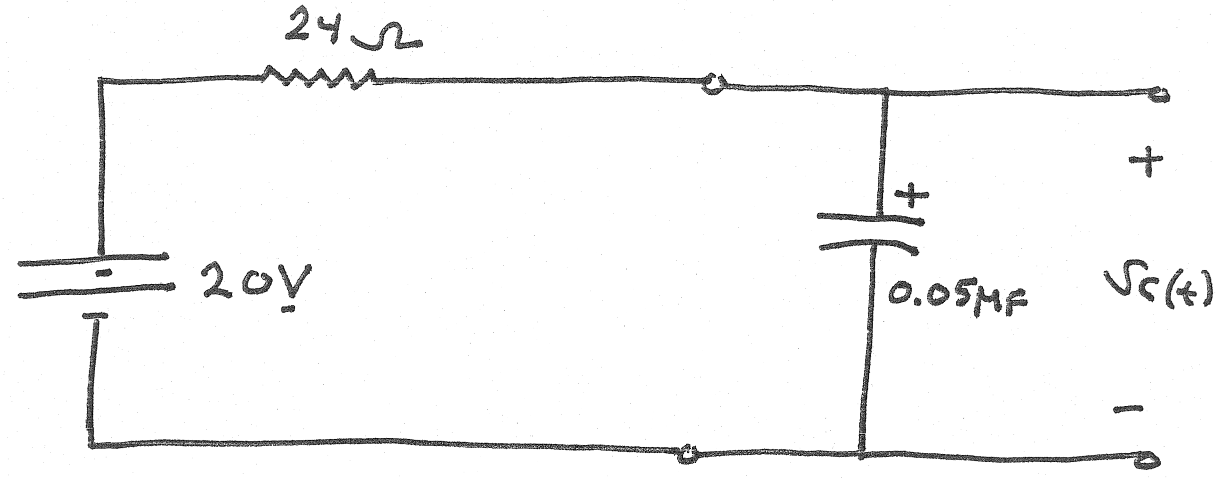


The Circuit can be simplified to



$$R_{TH} = 60\Omega \,||\, 200\Omega \,||\, 50\Omega = 24\Omega$$

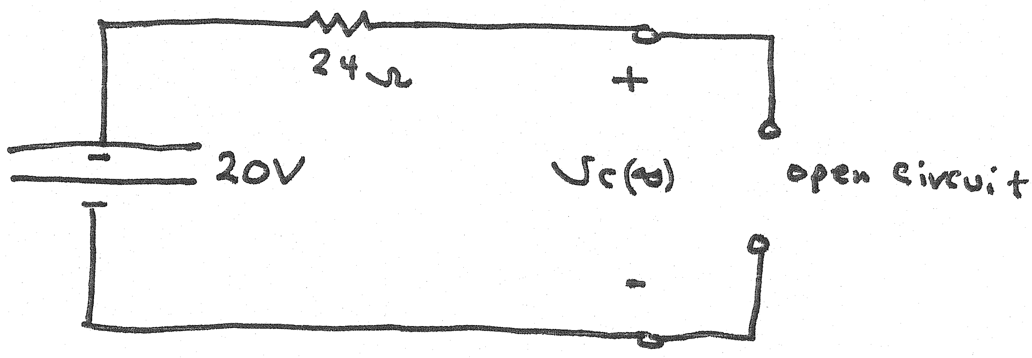
$$V_{TH} = \frac{50\Omega \,||\, 200\Omega}{50\Omega \,||\, 200\Omega + 60\Omega} \cdot 50 = 20V$$



$$v_c(t) = v_c(\infty) + \left[v_c(0^+) - v_c(\infty)\right] e^{-t/\tau} \; ; \; t > 0$$

1) $v_c(0^+) = v_c(0^-) = 100V$

2) To find $v_c(\infty)$



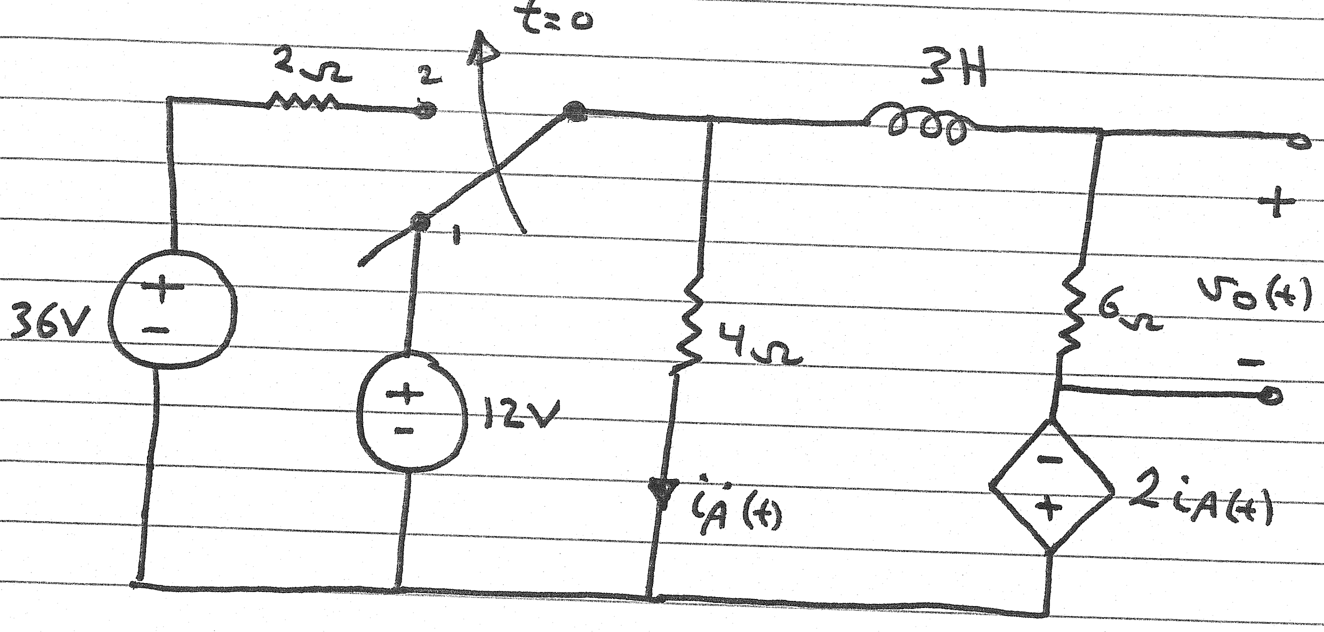
$\therefore \; v_c(\infty) = 20V$

3) $\tau = R_{TH}\, C = 1.2 \text{ sec}$

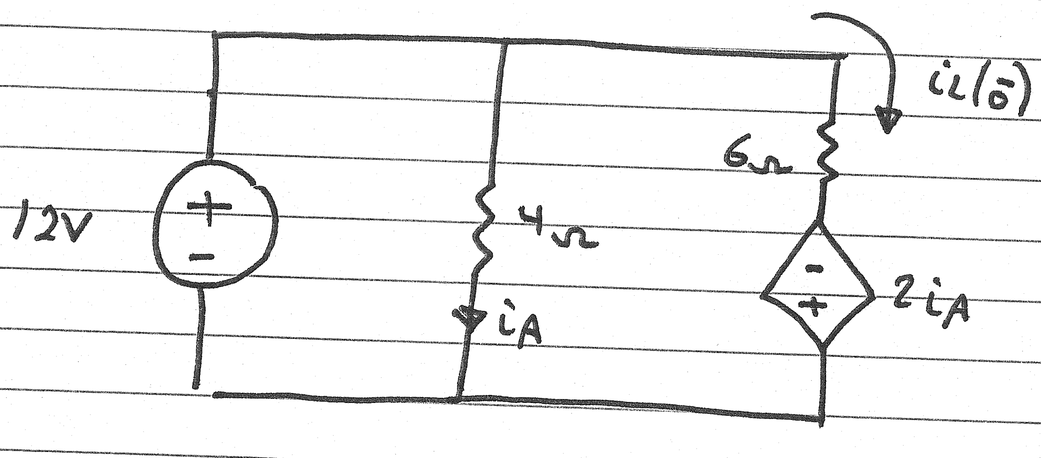
$$v_c(t) = 20 + (100 - 20)\, e^{-t/1.2} \; \underline{V}, \; t \geq 0$$

$$v_c(t) = 20 + 80\, e^{-t/1.2} \; \underline{V}, \quad t \geq 0$$

Find $v_o(t)$ for $t > 0$



For $t < 0$; $t = 0^-$



$$i_A = \frac{12}{4} = 3A$$

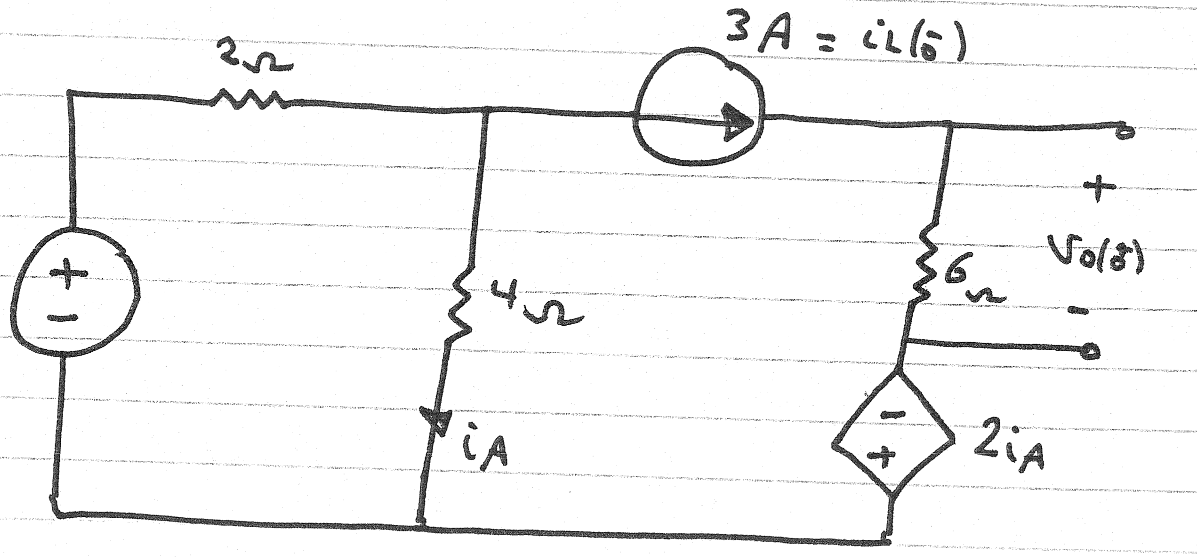
KVL :

$$-12 + 6\, i_L(0^-) - 2 i_A = 0$$

$$\therefore\ i_L(0^-) = 3A$$

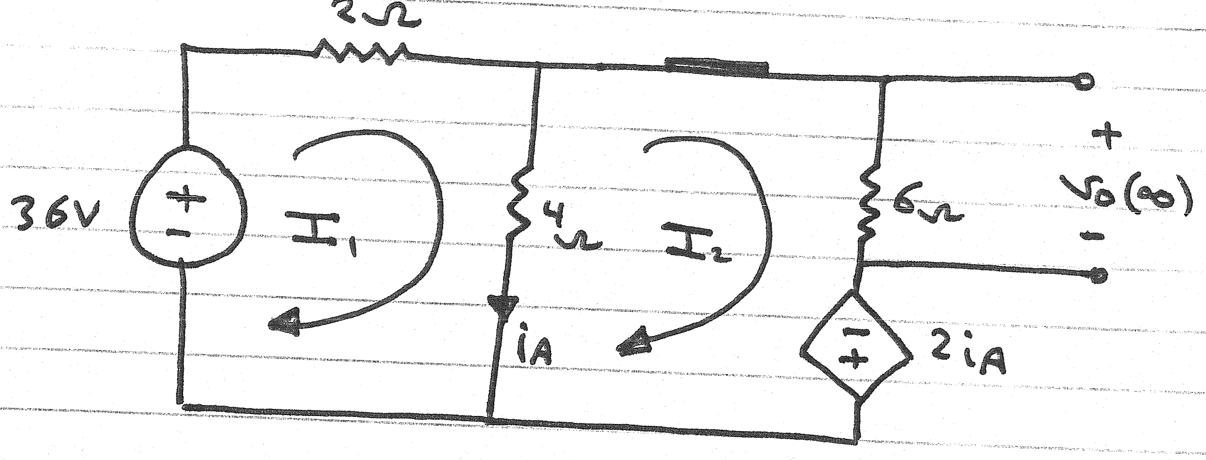
$$v_o(t) = v_o(\infty) + \left[v_o(0^+) - v_o(\infty)\right] e^{-t/\tau} \qquad t > 0$$

<u>To find $v_o(0^+)$</u>



$$v_o(0^+) = (3A)(6\Omega) = 18V$$

To find $v_o(\infty)$



KVL for mesh ① :

$$36 = 6I_1 - 4I_2$$

KVL for mesh ② :

$$2i_A = -4I_1 + 10I_2$$

$$i_A = I_1 - I_2$$

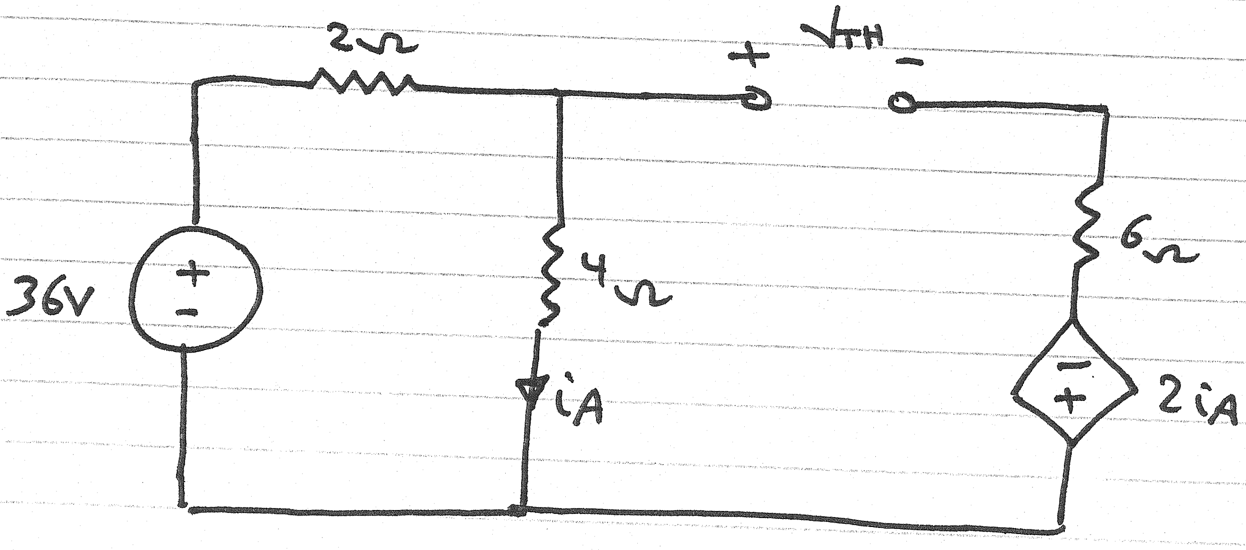
Solving for I_2 ; we get $I_2 = \frac{36}{8}$ A

$$v_o(\infty) = (6\Omega)\left(\frac{36}{8}A\right) = 27V$$

$$\text{To find } \tau = \frac{L}{R_{TH}}$$

To find R_{TH}

$$R_{TH} = \frac{V_{TH}}{I_{SC}}$$

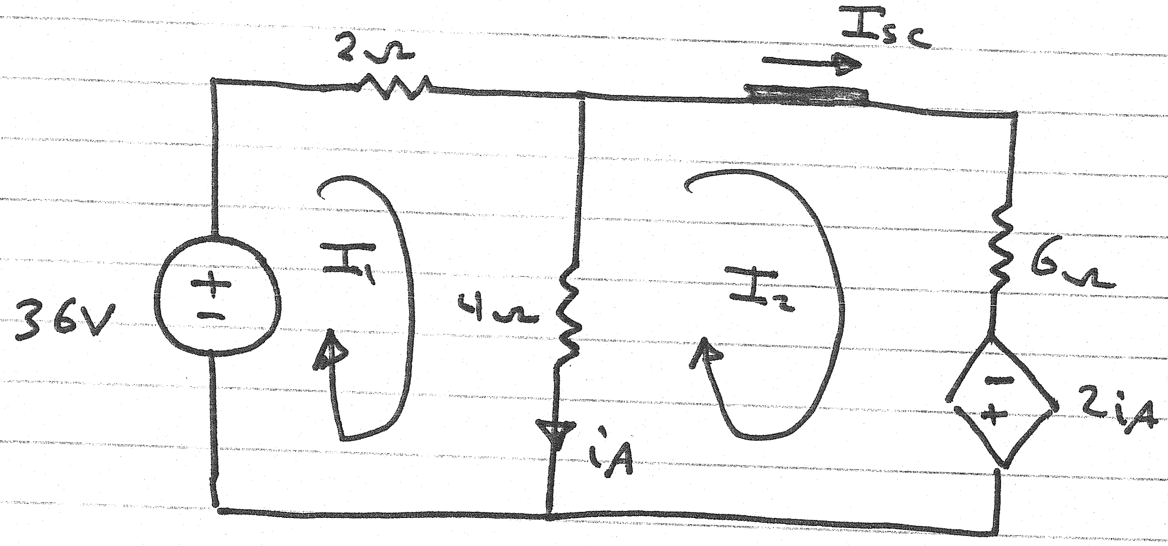


$$V_{TH} = 4\Omega\, i_A + 2\, i_A = 6\, i_A$$

$$i_A = \frac{36}{6} = 6A$$

$$\therefore V_{TH} = 36V$$

To find I_{sc}



KVL for mesh 1 :

$$36 = 6 I_1 - 4 I_2$$

KVL for mesh 2 :

$$2 i_A = -4 I_1 + 10 I_2$$

$$i_A = I_2 - I_1$$

Solving for $I_2 = I_{sc} = \frac{36}{8} A$

$$\therefore R_{TH} = \frac{V_{TH}}{I_{sc}} = 8\,\Omega$$

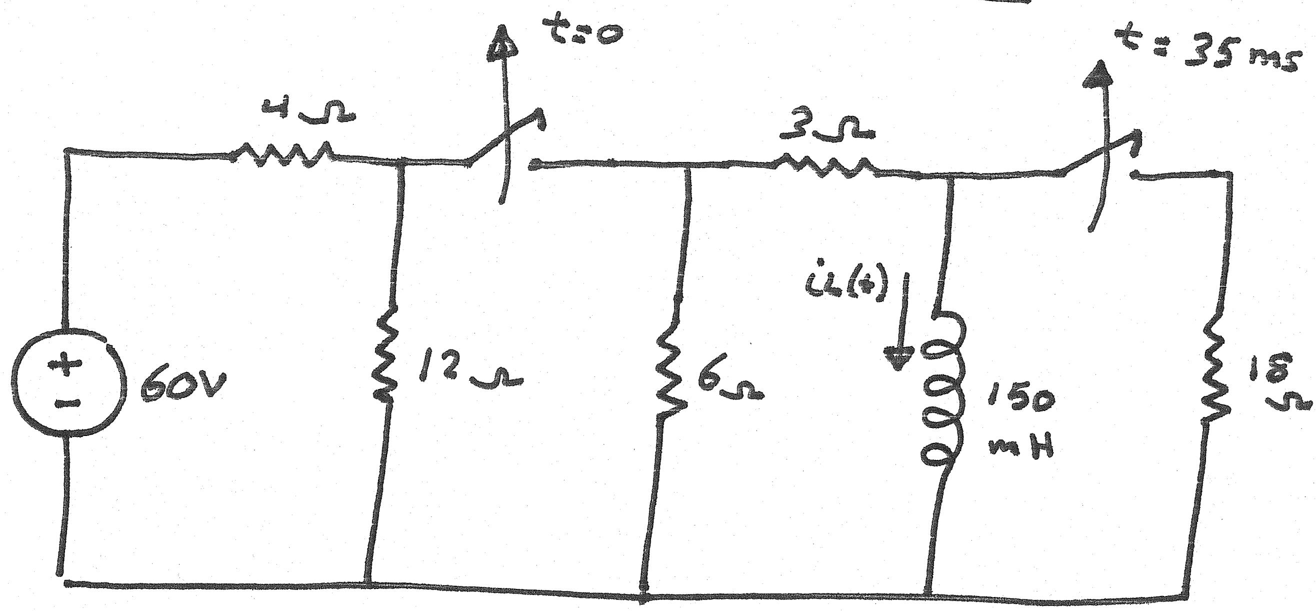
$$\therefore \tau = \frac{L}{R_{TH}} = \frac{3}{8} \text{ sec}$$

$$\therefore v_o(t) = v_o(\infty) + [v_o(0^+) - v_o(\infty)]\, e^{-t/\tau} \quad t>0$$

$$\therefore v_o(t) = 27 + [18 - 27]\, e^{-t/\tau} \quad t>0$$

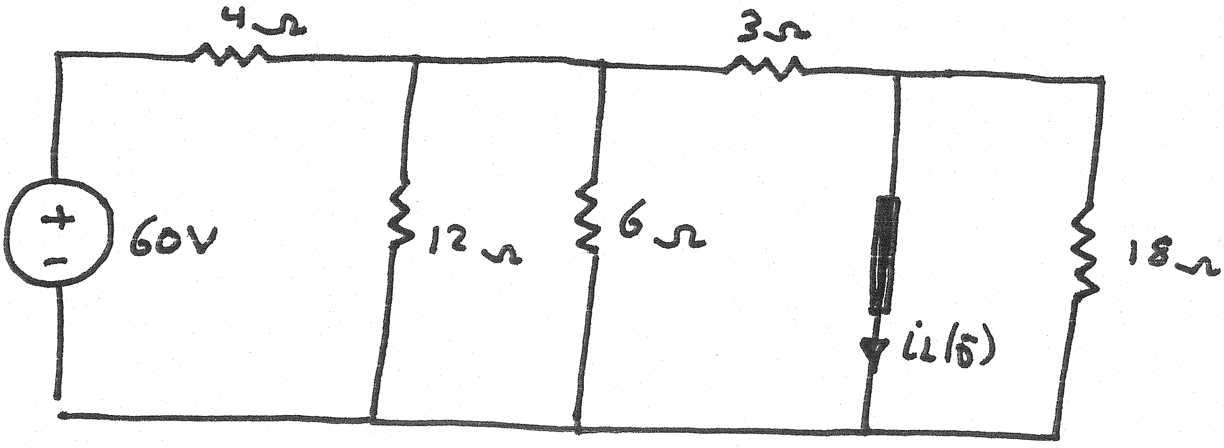
$$\therefore v_o(t) = (27 - 9\, e^{-\frac{8}{3}t}) \ \underline{V} \quad \text{for } t>0$$

Sequential Switching



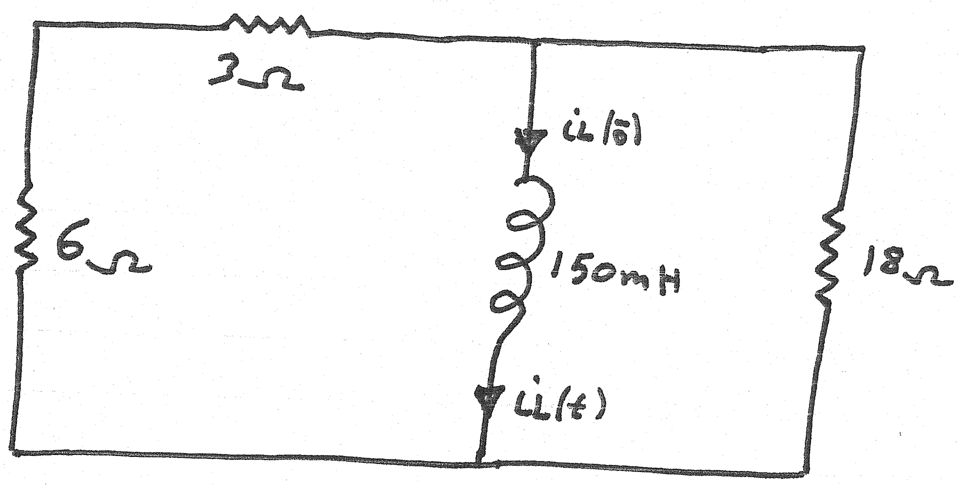
Find $i_L(t)$ for $t > 0$

1) For $t < 0$; $t = 0^-$



$i_L(0^-) = 6A$

2) For $35ms \geq t \geq 0$



Source-free circuit

$$\therefore \quad i_L(t) = A e^{-t/\tau_1} \quad , \quad t \geq 0$$

$$\tau_1 = \frac{L}{R_{TH_1}}$$

$$R_{TH_1} = 18\Omega \,||\, (6\Omega + 3\Omega) = 6\Omega$$

$$\therefore \quad \tau_1 = \frac{150mH}{6\Omega} = 25\ ms$$

To find A

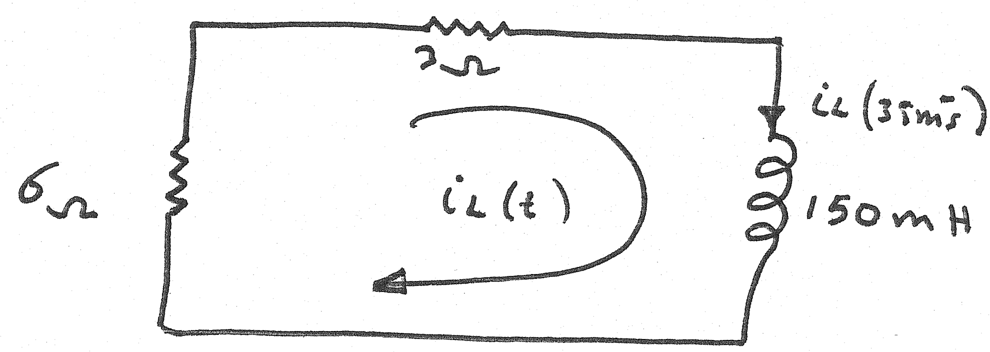
$$i(0^+) = A = i_L(0^-) = 6\underline{\underline{A}}$$

$$\therefore \quad i_L(t) = 6 e^{-40t} \underline{A} \quad , \quad 0 \leq t \leq 35ms$$

at $t = 35ms$

$$i_L(35ms) = 1.48\ \underline{A}$$

2) For $t > 35ms$



Source free Circuit

$$\therefore \quad i_L(t) = A\, e^{\frac{-(t-35ms)}{\tau_2}}$$

$$\tau_2 = \frac{L}{R_{TH2}}$$

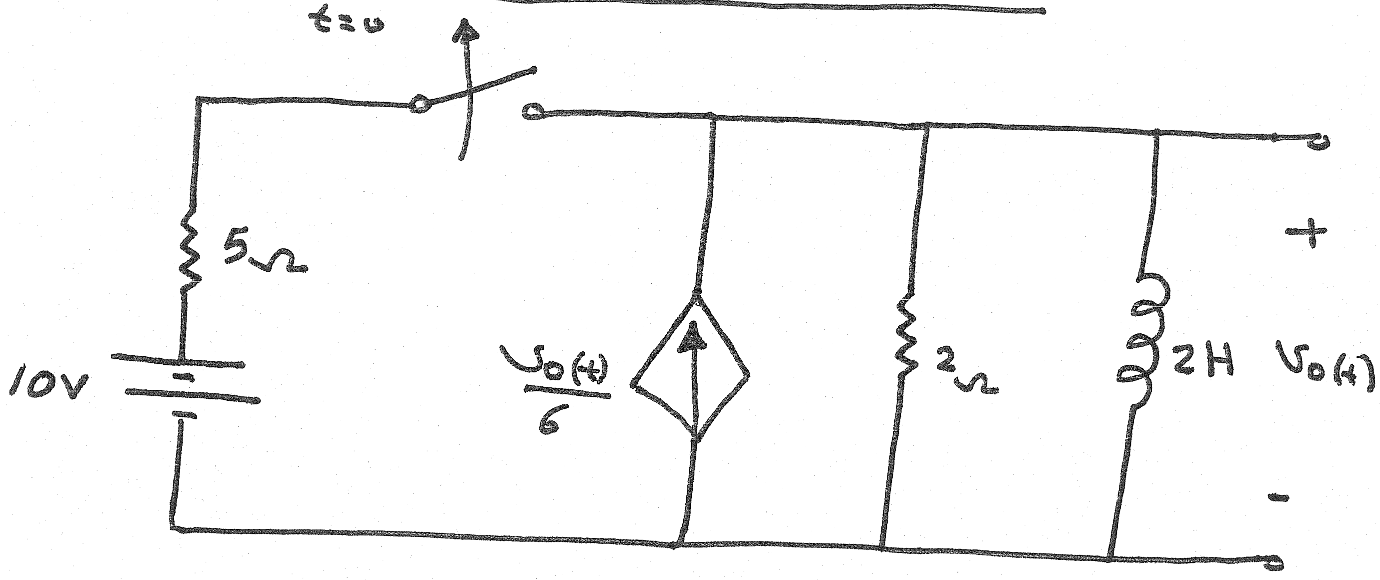
$$R_{TH2} = 6\Omega + 3\Omega = 9\Omega$$

$$\therefore \tau_2 = 16.67\ ms$$

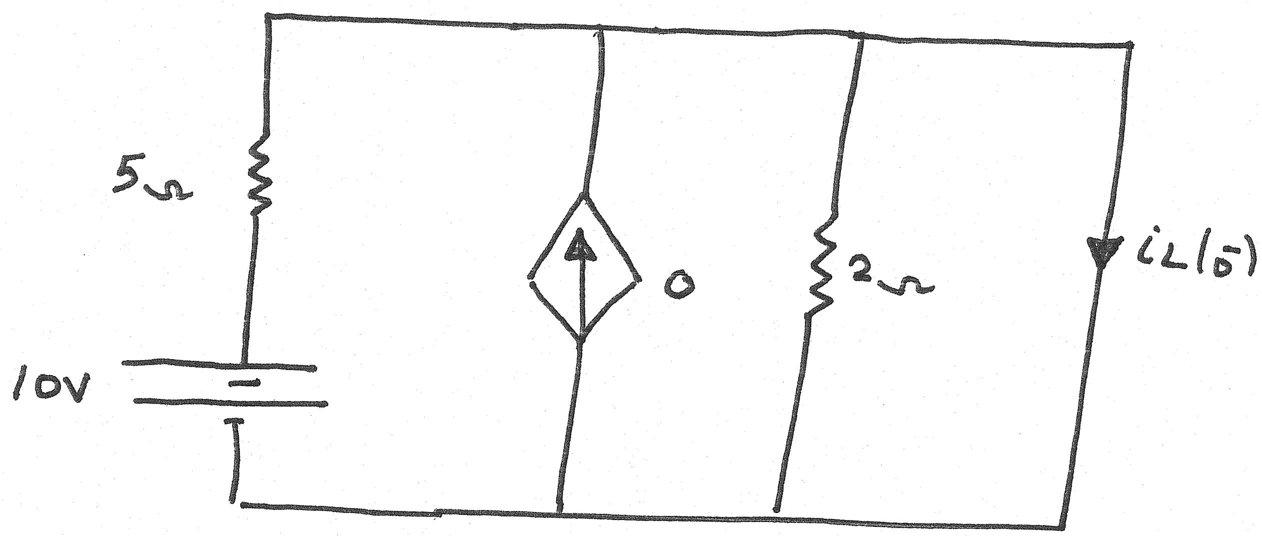
$$\therefore \quad i_L(t) = 1.48\, e^{\frac{-(t-35ms)}{16.67ms}}$$

$$i_L(t) = 1.48\, e^{-60(t-0.035)}\ \underline{A}\ ;\ t>35ms$$

Circuit with dependent sources

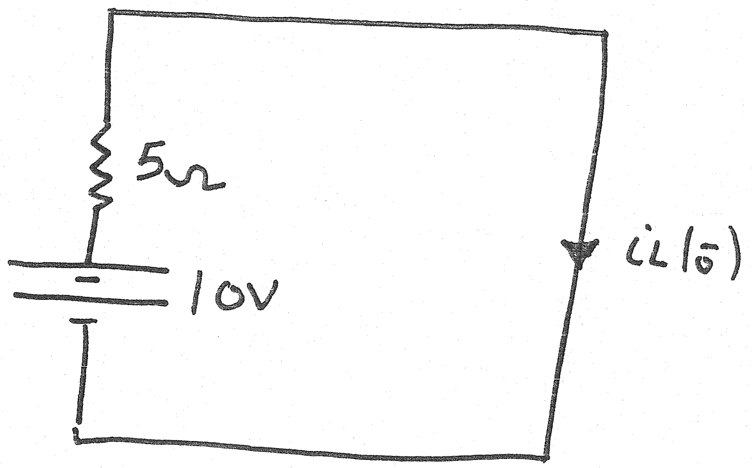


1) for $t < 0$; $t = 0^-$



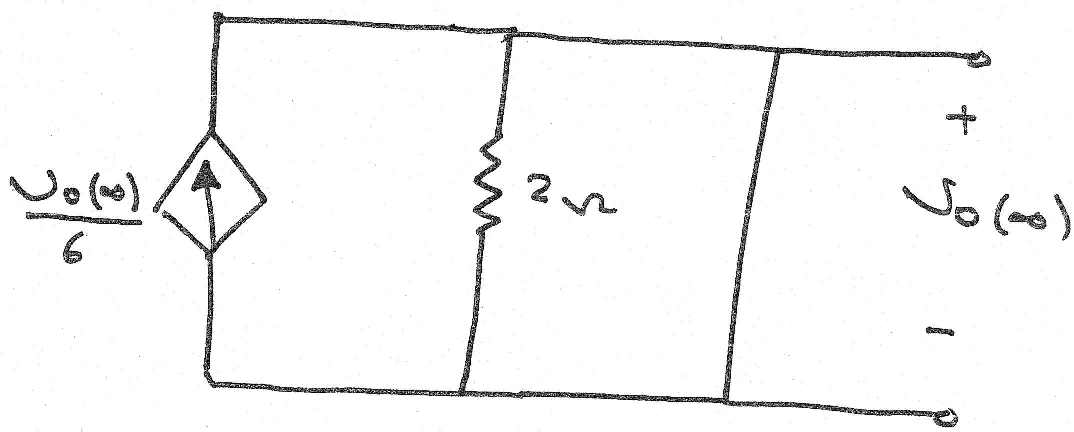
$\frac{v_o}{6} = 0 \longrightarrow$ open circuit

$2\Omega \| 0 = 0 \longrightarrow$ 2Ω $\longrightarrow$ open circuit



$\therefore\ i_L(0^-) = 2\underline{A}$

2) At $t = \infty$

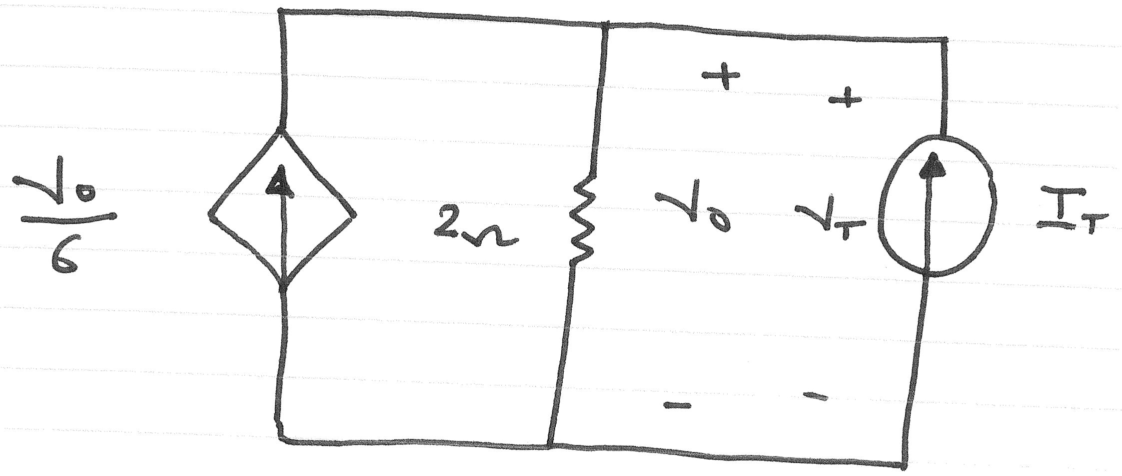


$\therefore\ v_o(\infty) = 0$

3) To find τ

$$\tau = \frac{L}{R_{TH}}$$

$$R_{TH} = \frac{V_T}{I_T}$$



KCL :

$$\frac{V_o}{6} + I_T = \frac{V_o}{2}$$

$$V_o = V_T$$

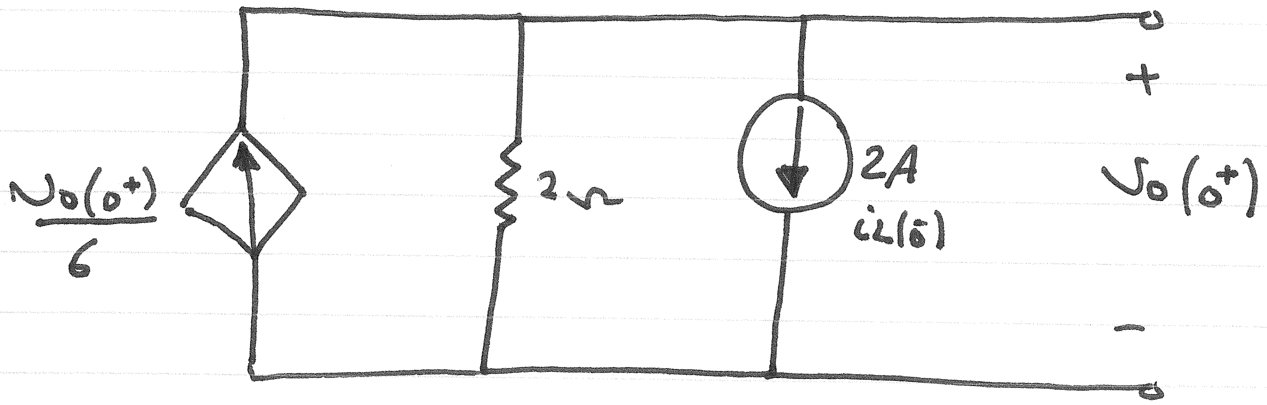
$$\therefore \frac{V_T}{I_T} = 3\,\Omega = R_{TH}$$

$$\therefore \tau = \frac{L}{R_{TH}} = \frac{2}{3}\text{ sec}$$

$$V_o(t) = V_o(\infty) + \left[V_o(0^+) - V_o(\infty)\right] e^{-t/\tau} \quad ; t>0$$

$$\therefore V_o(t) = -6\,e^{-1.5t} \text{ A} \quad ; \quad t>0$$

4) at $t = 0^+$



KCL :

$$\frac{v_o(0^+)}{6} = 2 + \frac{v_o(0^+)}{2}$$

$$\therefore \; v_o(0^+) = -6 \underline{V}$$

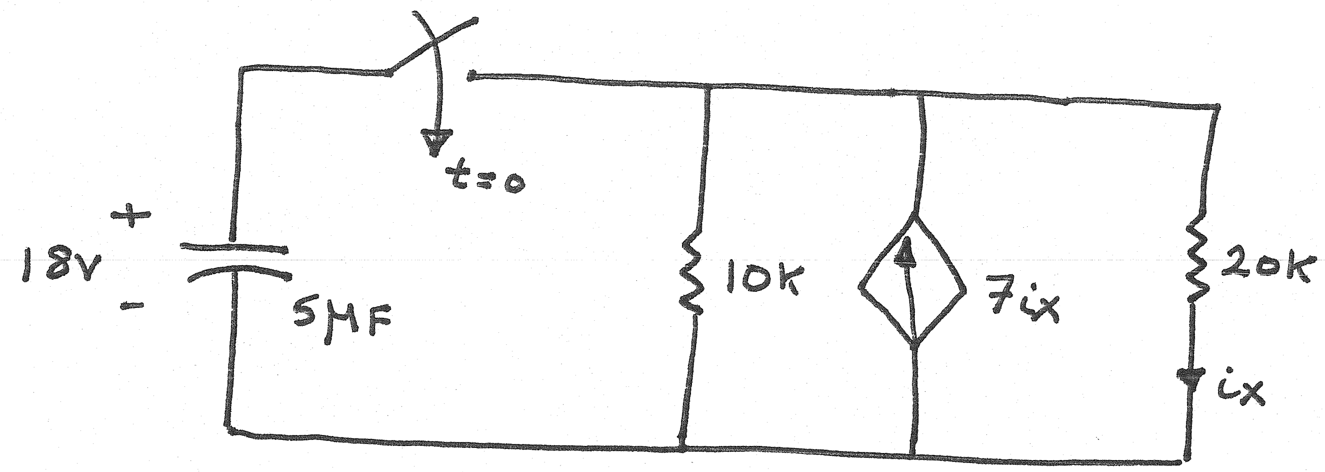
Now

$$v_o(t) = v_o(\infty) + \left[v_o(0^+) - v_o(\infty)\right] e^{-t/\tau} \;,\; t > 0$$

$$\therefore \; v_o(t) = -6 \, e^{-t/\tau} \quad \underline{A} \;;\; t > 0$$

$$\therefore \; v_o(t) = -6 \, e^{-1.5t} \quad A \;;\; t > 0$$

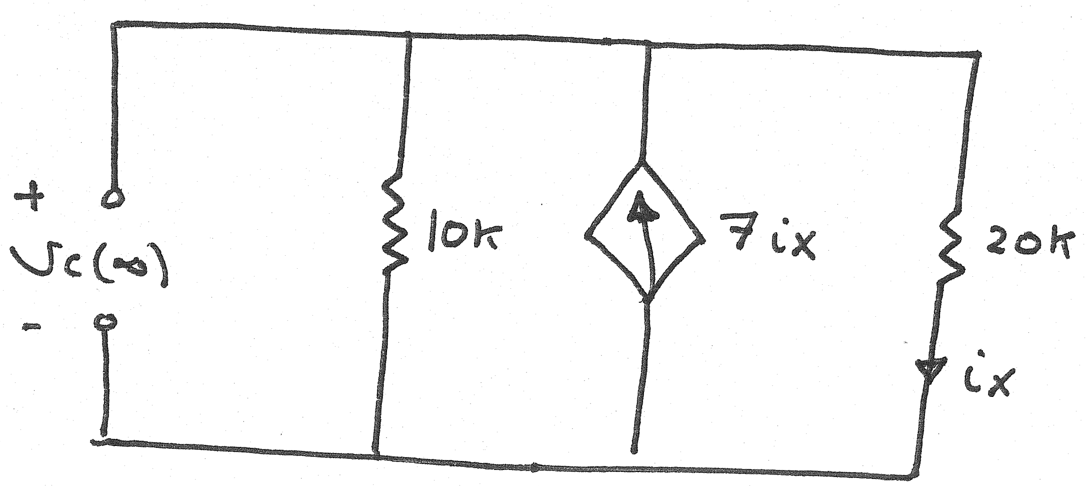
Unbounded Response



Find $v_c(t)$ for $t > 0$

1) $v_c(0^+) = v_c(0^-) = 18\,\underline{V}$

2) To find $v_c(\infty)$



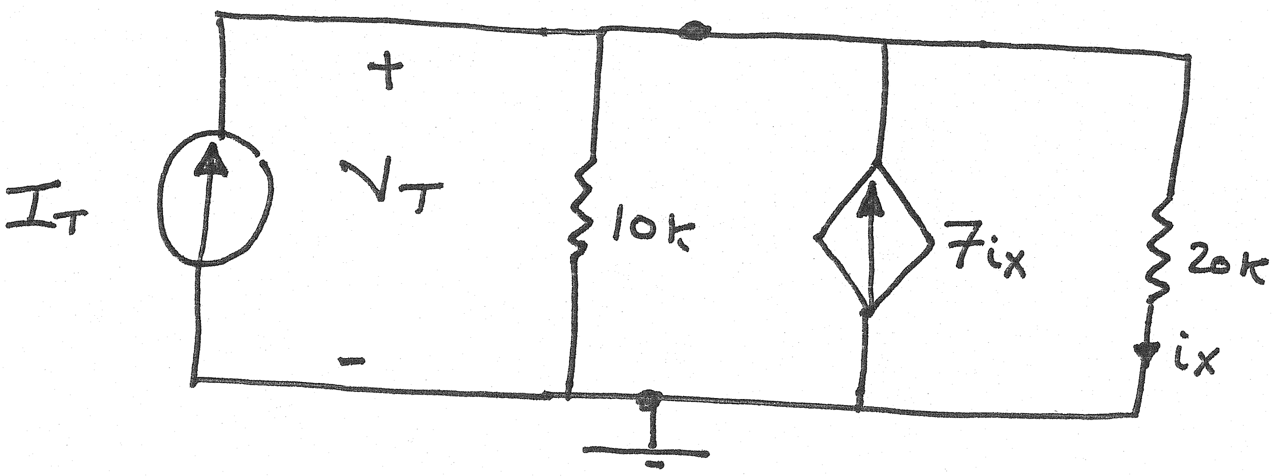
Since the Circuit is dead

$\therefore v_c(\infty) = 0$

3) To find τ

$$\tau = R_{TH}\,C$$

$$R_{TH} = \frac{V_T}{I_T}$$



KCL :

$$I_T + 7\,i_x = \frac{V_T}{10k} + \frac{V_T}{20k}$$

$$i_x = \frac{V_T}{20k}$$

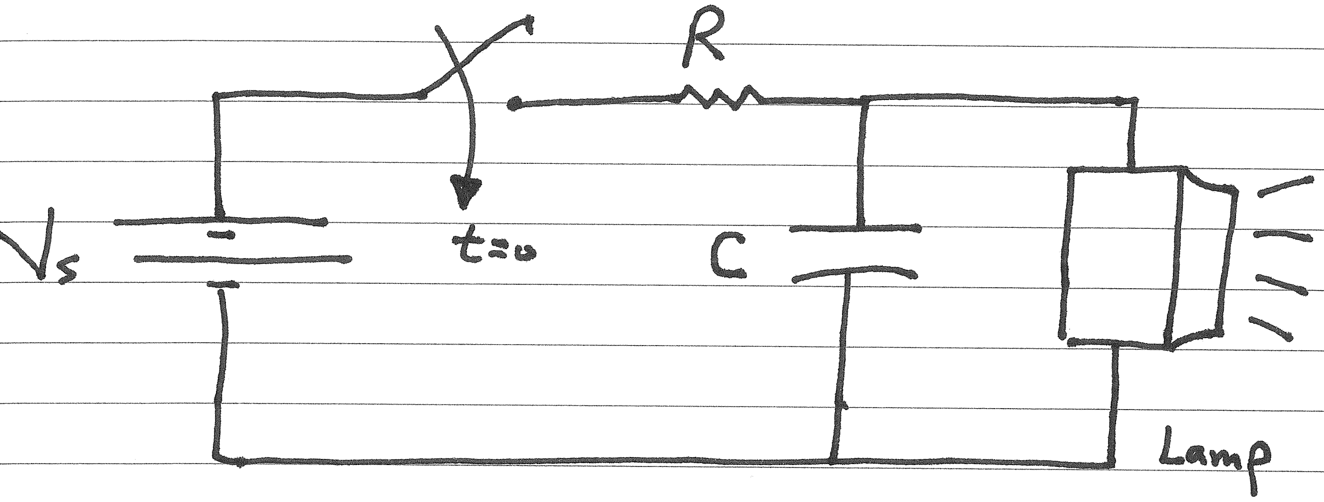
$$\therefore \frac{V_T}{I_T} = -5k = R_{TH} \quad *$$

$$\therefore \tau = R_{TH}\,C = -25\,ms$$

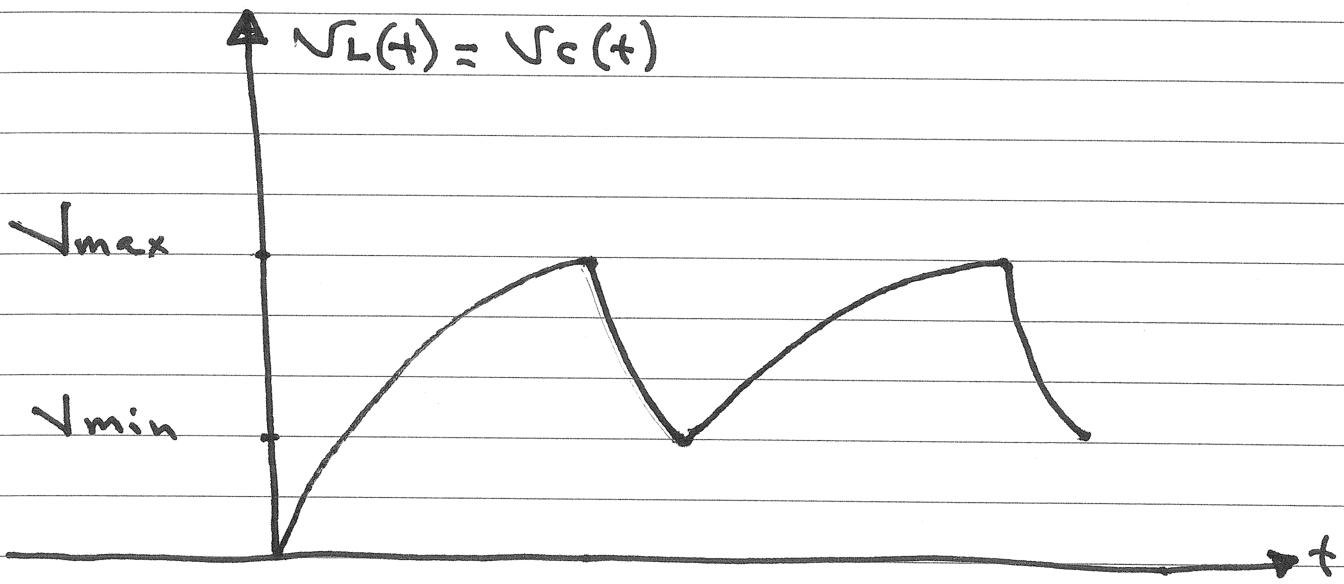
$$\therefore V_c(t) = 18\,e^{+40t}\ \underline{V} \qquad \text{for } t \geqslant 0$$

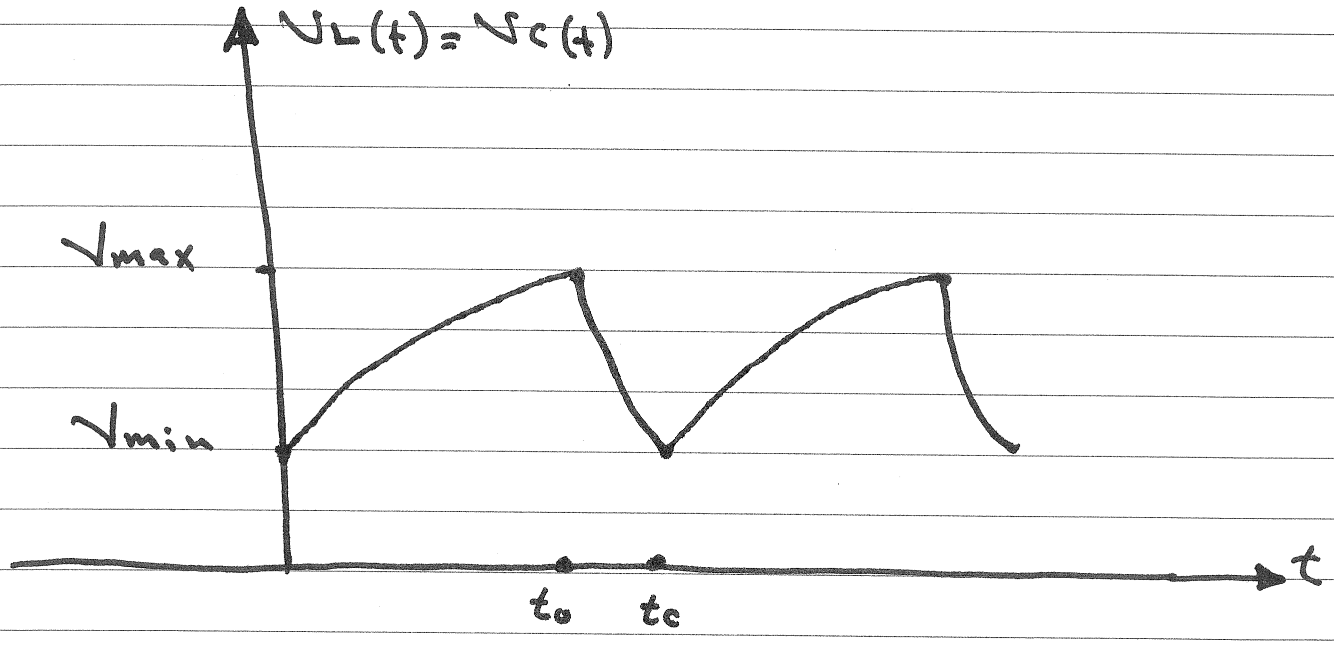
Practical Perspective

A Flashing Light Circuit



- The Lamp starts to Conduct whenever the Lamp voltage reaches V_{max}
- During the time the Lamp Conducts, it Can be modeled as R_L
- The Lamp will Continue to Conduct untill the Lamp voltage drops to V_{min}
- During the time the lamp is not Conducting, it can be modeled as open Circuit.





1) In the interval $t_o > t > 0$

$$V_L(t) = V_s + (V_{min} - V_s)\, e^{-t/\tau_1}$$

$$\tau_1 = RC \quad , \quad R_L = \infty$$

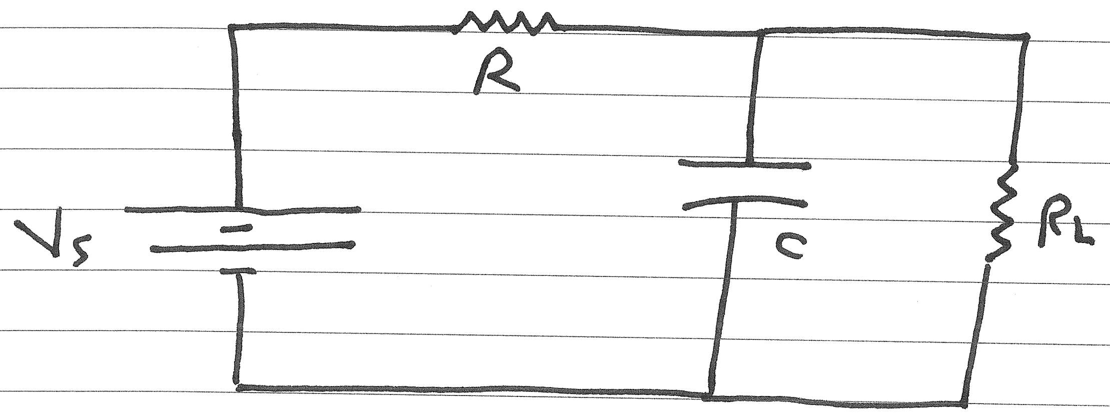
$$v_L(t) = V_s + (V_{min} - V_s)\, e^{-t/\tau_1}$$

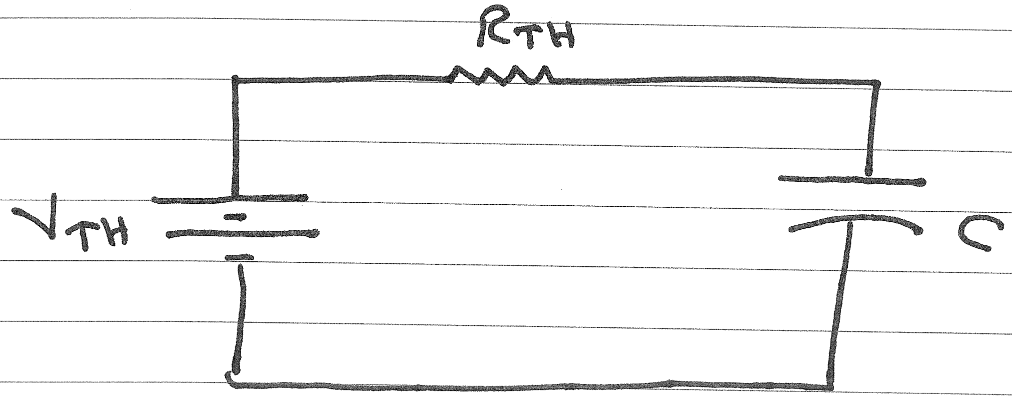
at $t = t_o$; $v_L(t_o) = V_{max}$

$$V_{max} = V_s + (V_{min} - V_s)\, e^{-t/RC}$$

$$\therefore \; t_o = RC \ln \frac{V_{min} - V_s}{V_{max} - V_s}$$

2) In the interval $t_c > t > t_o$





$$R_{TH} = R \parallel R_L$$

$$V_{TH} = \frac{R_L}{R_L + R_{TH}} V_s$$

$$V_L(t) = V_{TH} + (V_{max} - V_{TH})\, e^{\frac{-(t-t_0)}{\tau_2}}$$

$$\tau_2 = R_{TH}C = \frac{R R_L}{R_L + R}\, C$$

at $t = t_c$; $V_L(t_c) = V_{min}$

$$V_{min} = V_{TH} + (V_{max} - V_{TH})\, e^{\frac{-(t_c-t_0)}{\tau_2}}$$

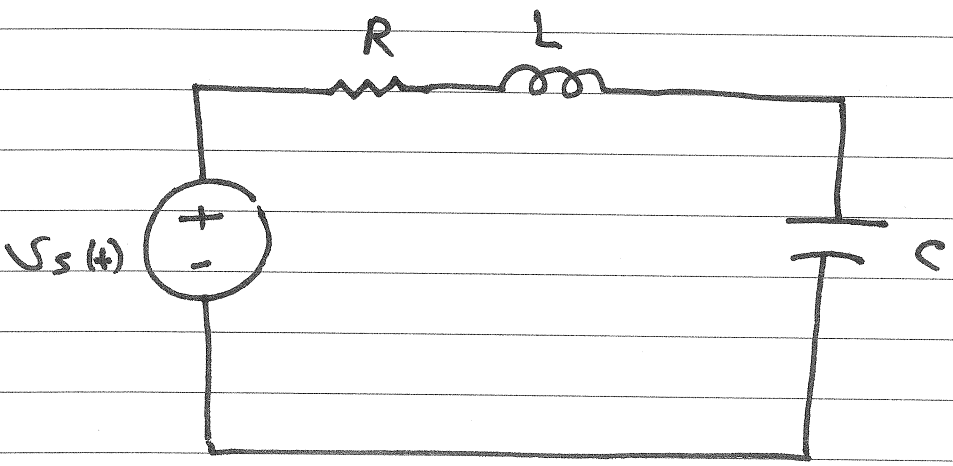
$$\therefore \quad t_c - t_0 = \frac{R R_L}{R + R_L}\, C \ln \frac{V_{max} - V_{TH}}{V_{min} - V_{TH}}$$

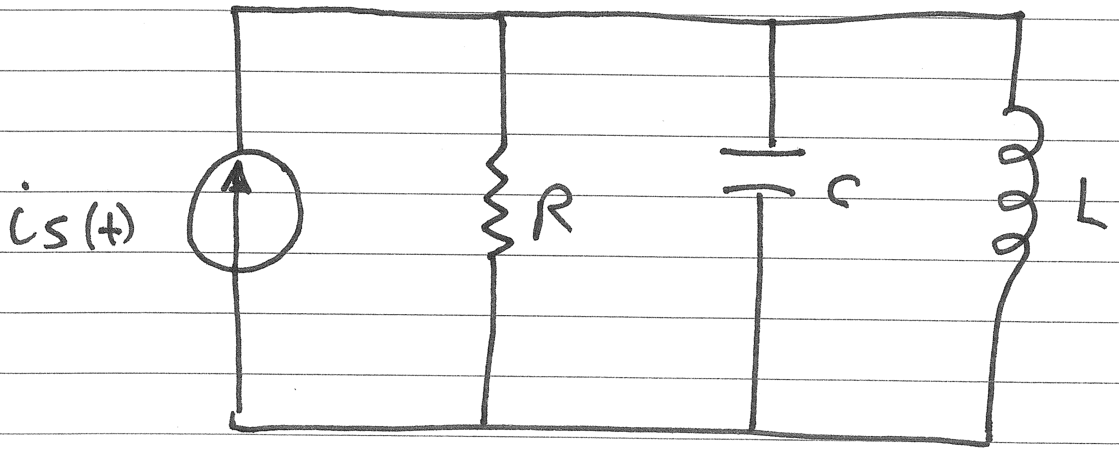
Chapter 8

Natural and Step Response of RLC Circuits

What is a 2nd order Circuit ?

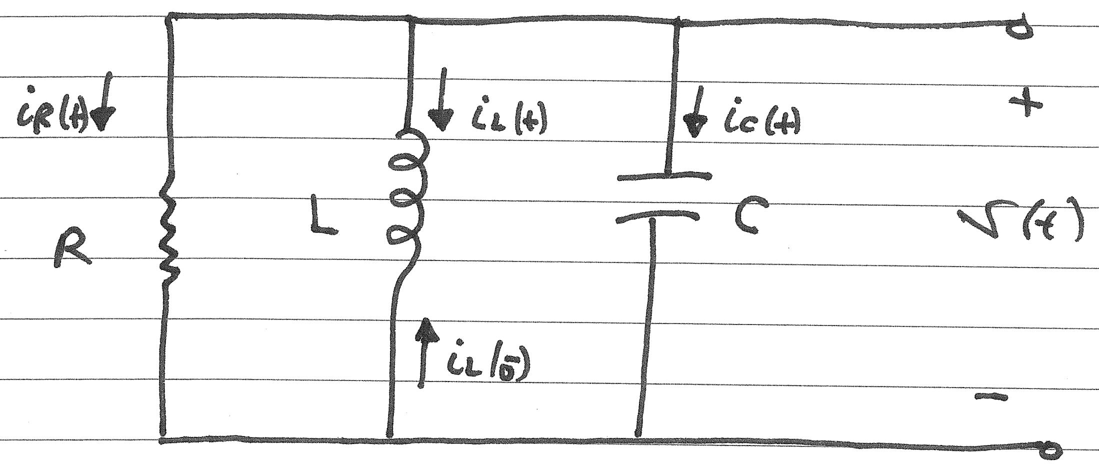
A second-order circuit is characterized by a second-order differential equation.





Natural Response of Parallel RLC Circuit

For $t > 0$



$$v_C(0^-) = 0 \quad ; \quad i_L(0^-) = 10\,A$$

KCL :

$$i_R(t) + i_L(t) + i_C(t) = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt - i_L(0^-) + C\frac{dv(t)}{dt} = 0$$

$$\frac{v(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt + C\frac{dv(t)}{dt} = i_L(0^-) \quad — (1)$$

Differentiate (1)

$$C\frac{d^2v(t)}{dt^2} + \frac{1}{R}\frac{dv(t)}{dt} + \frac{1}{L}v(t) = 0$$

second-order homogeneouse differential equation

$$\therefore \; v(t) = A e^{st} \qquad \text{for } t > 0$$

$$C A s^2 e^{st} + \frac{1}{R} s A e^{st} + \frac{1}{L} A e^{st} = 0$$

$$A e^{st} \left(C s^2 + \frac{1}{R} s + \frac{1}{L} \right) = 0$$

$$\therefore \; C s^2 + \frac{1}{R} s + \frac{1}{L} = 0$$

Characteristic equation

$$s_{1,2} = \frac{-b \mp \sqrt{b^2 - 4ac}}{2a}$$

$$s_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$s_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

Let $w_o = \dfrac{1}{\sqrt{LC}}$

$w_o \equiv$ resonant frequency

and

$$\alpha = \frac{1}{2RC}$$

$\alpha \equiv$ damping Coefficient

$\therefore$

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_o^2}$$

1) If $\alpha > \omega_o$, the solutions are real, unequal and the response is termed overdamped.

2) If $\alpha < \omega_o$, the solutions are complex conjugates and the response is termed Underdamped.

3) If $\alpha = \omega_o$, the solutions are real and equal and the response is termed Critically damped.

1) The overdamped case

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

If $\alpha > \omega_0$

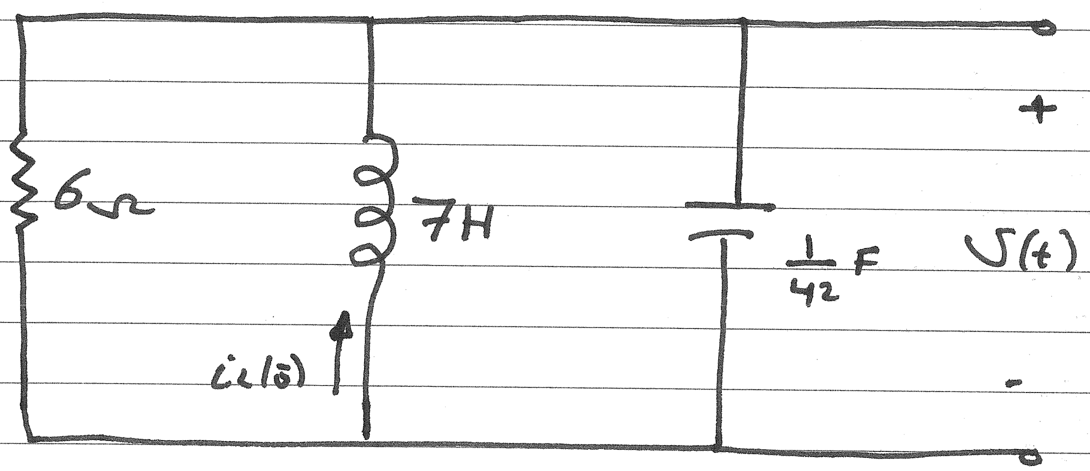
$$\therefore \quad S_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \quad < 0$$

$$S_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} \quad < 0$$

S_1, S_2 are real, unequal

$$\therefore \quad v(t) = A_1 e^{S_1 t} + A_2 e^{S_2 t} \qquad t > 0$$

Overdamped Parallel RLC



$v_C(0^-) = 0$, and $i_L(0^-) = 10\,A$

$$\alpha = \frac{1}{2RC} = 3.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6} = 2.45$$

$\therefore \alpha > \omega_0$ overdamped

$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -1$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -6$$

$\therefore v(t) = A_1 e^{-t} + A_2 e^{-6t}$ for $t > 0$

To find A_1, and A_2, we need

$v(0^+)$ and $\frac{dv(0^+)}{dt}$

For $t > 0$

$$\frac{v(t)}{R} + \frac{1}{L}\int_{0^-}^{t} v(t)\,dt + C\frac{dv(t)}{dt} = i_L(0^-)$$

at $t = 0^+$

$$\frac{v(0^+)}{R} + \frac{1}{L}\int_{0^-}^{0^+} v(t)\,dt + C\frac{dv(0^+)}{dt} = i_L(0^-)$$

$$v(0^+) = v_c(0^+) = v_c(0^-) = 0$$

$$\int_{0^-}^{0^+} v(t)\,dt = 0$$

$\therefore$ $$i_L(0^-) = C\frac{dv(0^+)}{dt}$$

$$\frac{dv(0^+)}{dt} = \frac{i_L(0^-)}{C} = 420$$

also $v(0^+) = 0 = v_c(0^-)$

$$v(t) = A_1 e^{-t} + A_2 e^{-6t} \quad \text{for } t>0$$

$$v(0^+) = A_1 + A_2 = v(0^-) = 0$$

$$\therefore A_1 + A_2 = 0 \qquad \text{——————} \quad (2)$$

$$v(t) = A_1 e^{-t} + A_2 e^{-6t}$$

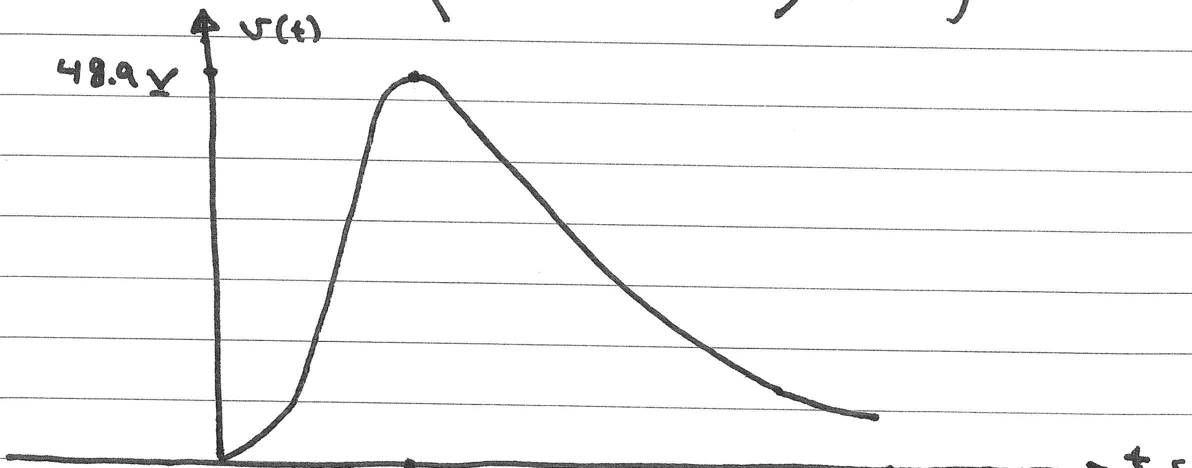
$$\frac{dv(t)}{dt} = -A_1 e^{-t} - 6A_2 e^{-6t}$$

$$\frac{dv(0^+)}{dt} = -A_1 - 6A_2 = 420 \quad \text{———} \quad (3)$$

Solving (2) and (3), we get

$$A_1 = 84 \quad \text{and} \quad A_2 = -84$$

$$\therefore v(t) = 84\left(e^{-t} - e^{-6t}\right) \text{ V} \quad \text{for } t>0$$



2) Critical Damping Case

$$\alpha = \omega_0$$

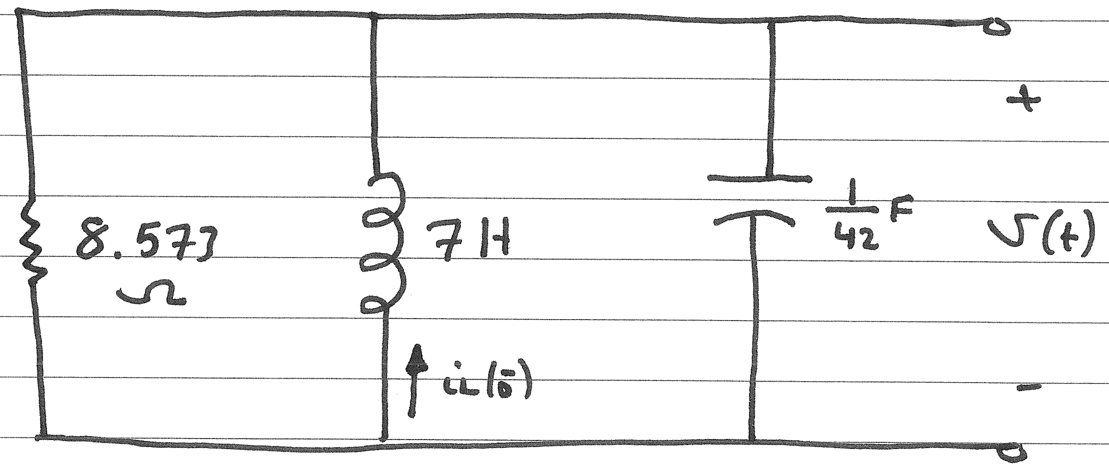
$$s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2} = -\alpha$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2} = -\alpha$$

$$s_1 = s_2 \qquad \text{real and equal}$$

$$\therefore \; v(t) = A_1 t e^{-\alpha t} + A_2 e^{-\alpha t} \quad \text{for } t > 0$$

Critical Damped Parallel RLC



$v_c(0^-) = 0$, and $i_L(0^-) = 10\,A$

$$\alpha = \frac{1}{2RC} = \sqrt{6}$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$$s_1 = -\sqrt{6}$$

$$s_2 = -\sqrt{6}$$

$$\therefore \quad v(t) = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t} \quad \text{for } t > 0$$

$$v(0^+) = 0$$

$$\frac{dv(0^+)}{dt} = 420$$

$$v(t) = A_1 t e^{-\sqrt{6}t} + A_2 e^{-\sqrt{6}t} \quad \text{for } t > 0$$

$$v(0^+) = A_2 = 0$$

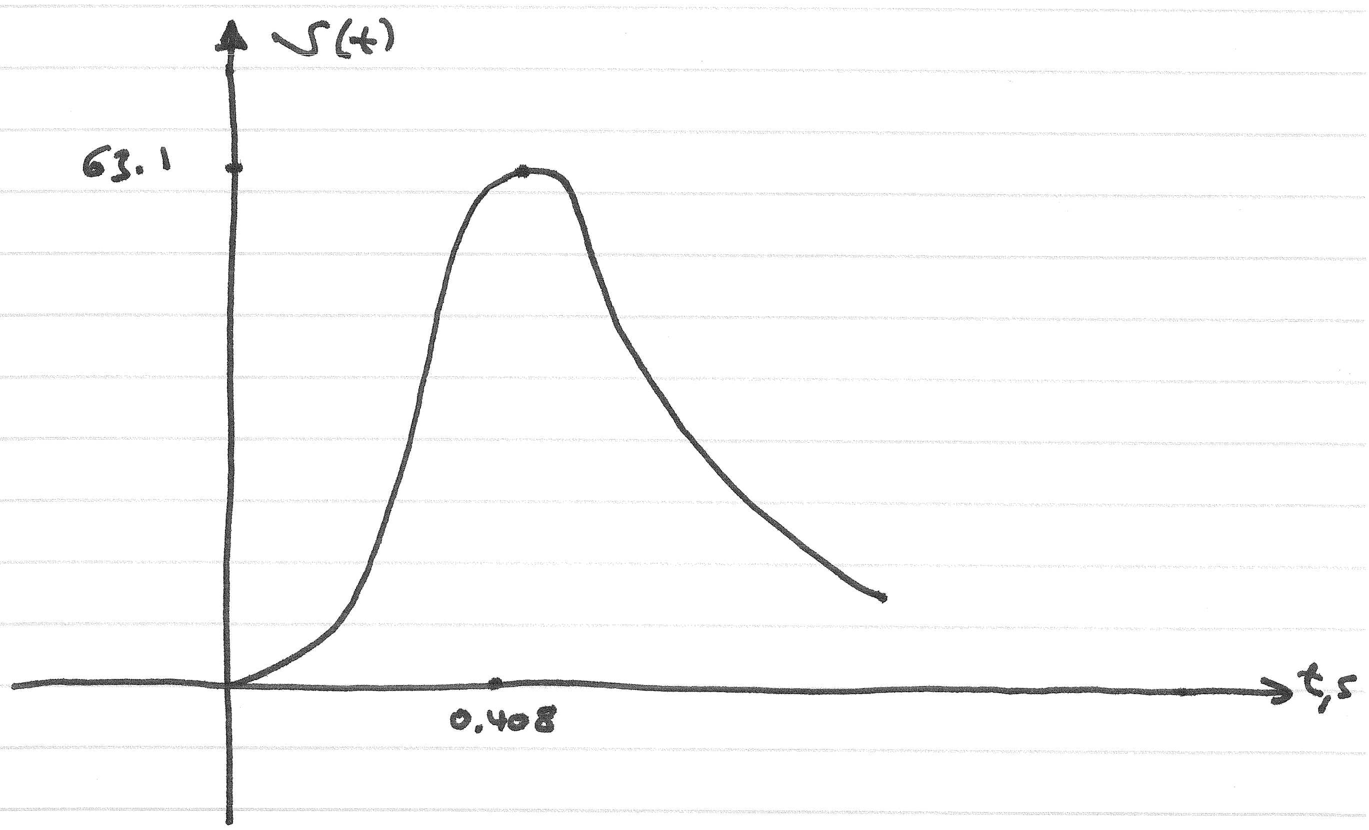
$$\therefore v(t) = A_1 t e^{-\sqrt{6}t} \quad \text{for } t > 0$$

$$\frac{dv(t)}{dt} = (A_1 t)(-\sqrt{6}) e^{-\sqrt{6}t} + A_1 e^{-\sqrt{6}t}$$

$$\frac{dv(0^+)}{dt} = 0 + A_1 = 420$$

$$\therefore A_1 = 420$$

$$\therefore v(t) = 420 t e^{-\sqrt{6}t} \text{ V} \quad \text{for } t > 0$$



3) The underdamped Case

$$\alpha < \omega_o$$

$$\therefore \quad \alpha^2 - \omega_o^2 < 0$$

$$\sqrt{\alpha^2 - \omega_o^2} = \sqrt{(-1)(\omega_o^2 - \alpha^2)}$$

$$\sqrt{\alpha^2 - \omega_o^2} = j\sqrt{\omega_o^2 - \alpha^2}$$

$$\sqrt{\alpha^2 - \omega_o^2} = j\,\omega_d$$

$\omega_d \equiv$ damped radian frequency

$$S_1 = -\alpha + \sqrt{\alpha^2 - \omega_o^2}$$

$$S_1 = -\alpha + j\,\omega_d$$

$$S_2 = -\alpha - j\,\omega_d$$

S_1, and S_2 are Complex Conjucate

$$S_1 = -\alpha + jW_d$$

$$S_2 = -\alpha - jW_d$$

$$\therefore\ v(t) = A_1 e^{(-\alpha + jW_d)t} + A_2 e^{(-\alpha - jW_d)t}$$

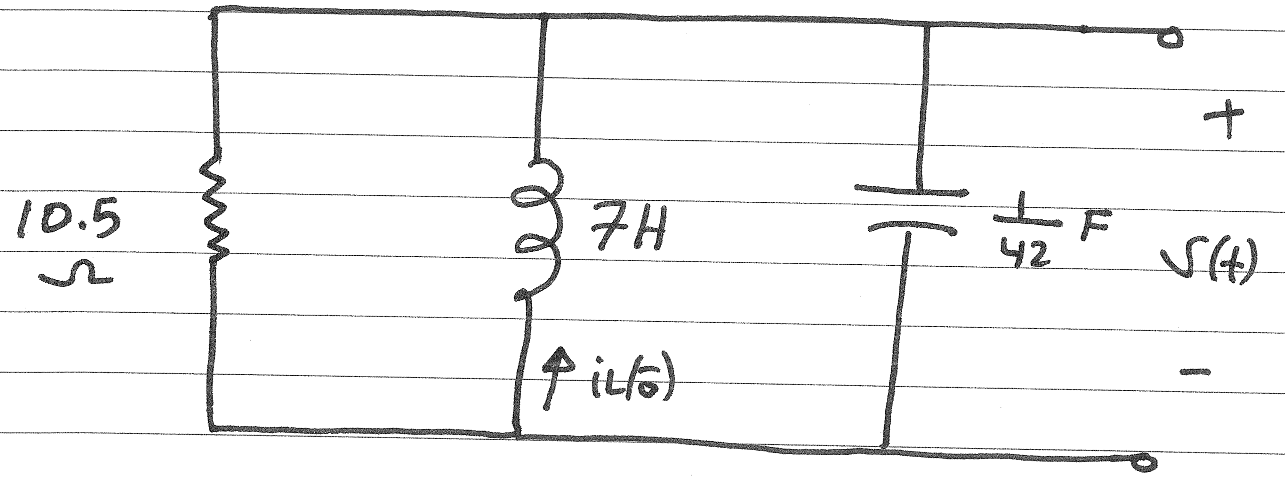
$$e^{jW_d t} = \cos W_d t + j\sin W_d t$$

$$e^{-jW_d t} = \cos W_d t - j\sin W_d t$$

$$v(t) = e^{-\alpha t}\left[(A_1 + A_2)\cos W_d t + j(A_1 - A_2)\sin W_d t\right]$$

$$\therefore\ v(t) = e^{-\alpha t}\left[B_1 \cos W_d t + B_2 \sin W_d t\right]$$

Underdamped Parallel RLC



$v_c(0^-) = 0$, and $i_L(0^-) = 10\,A$

$$\alpha = \frac{1}{2RC} = 2$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6} =$$

$\therefore \quad \alpha < \omega_0$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{2}$$

$$\therefore v(t) = e^{-\alpha t}\left(B_1 \cos \omega_d t + B_2 \sin \omega_d t\right) \quad \text{for } t > 0$$

$$\therefore v(t) = e^{-2t}\left(B_1 \cos\sqrt{2}\,t + B_2 \sin\sqrt{2}\,t\right) \quad \text{for } t > 0$$

$$v(0^+) = 0$$

$$\frac{dv(0^+)}{dt} = 420$$

$$v(t) = e^{-2t}\left(B_1 \cos\sqrt{2}\,t + B_2 \sin\sqrt{2}\,t\right)$$

$$v(0^+) = B_1 = 0$$

$$\therefore B_1 = 0$$

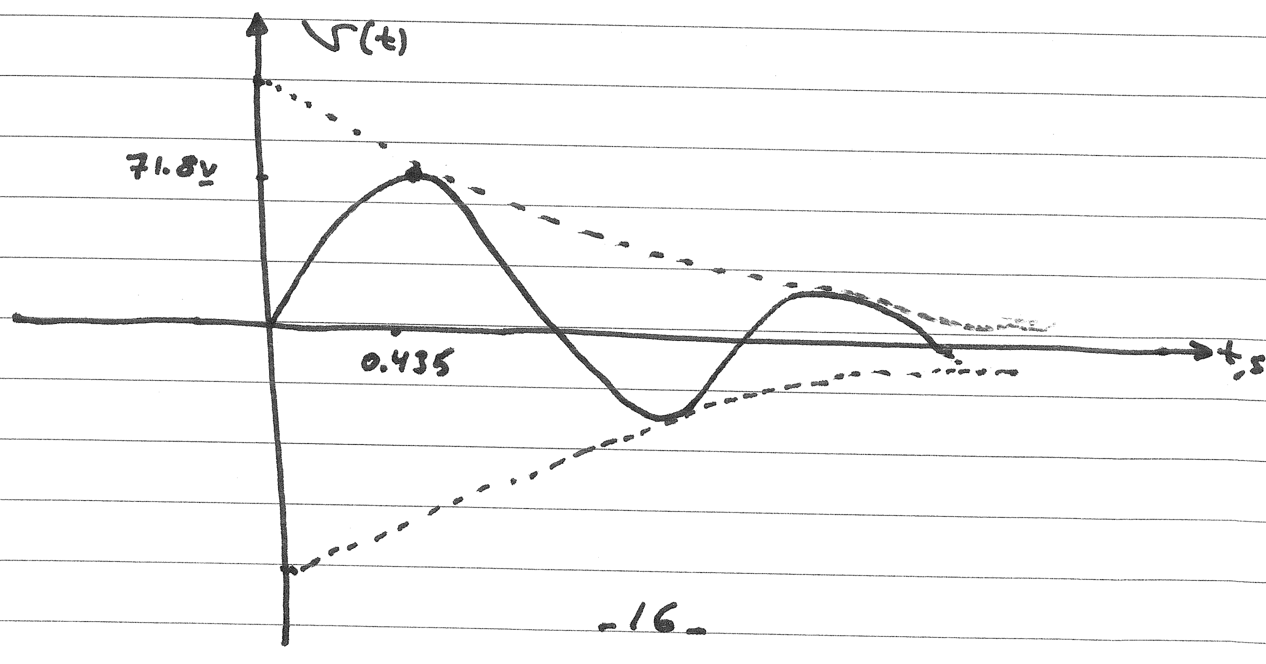
$$\therefore v(t) = e^{-2t} B_2 \sin\sqrt{6}\,t \ \text{V} \quad \text{for } t>0$$

$$\frac{dv(t)}{dt} = (B_2 e^{-2t})(\sqrt{2}\cos\sqrt{2}\,t) + (\sin\sqrt{2})(-2e^{-2t} B_2)$$

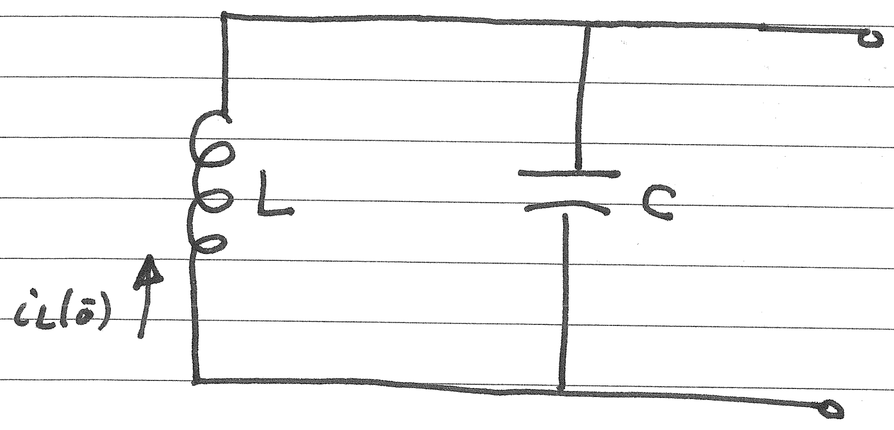
$$\frac{dv(0^+)}{dt} = \sqrt{2}\, B_2 + 0 = 420$$

$$\therefore B_2 = \frac{420}{\sqrt{2}}$$

$$\therefore v(t) = \frac{420}{\sqrt{2}} e^{-2t} \sin\sqrt{2}\,t \ \text{V} \quad \text{for } t>0$$



The Lossless LC Circuit



$L = 7H$, $C = \frac{1}{42} F$, and $R = \infty$

$$\alpha = \frac{1}{2RC} = 0$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{6}$$

$$\alpha < \omega_0$$

$$\omega_d = \sqrt{\omega_0^2 - \alpha^2} = \sqrt{6}$$

$\therefore \quad v(t) = \beta_1 \cos\sqrt{6}\, t + \beta_2 \sin\sqrt{6}\, t \qquad$ for $t > 0$

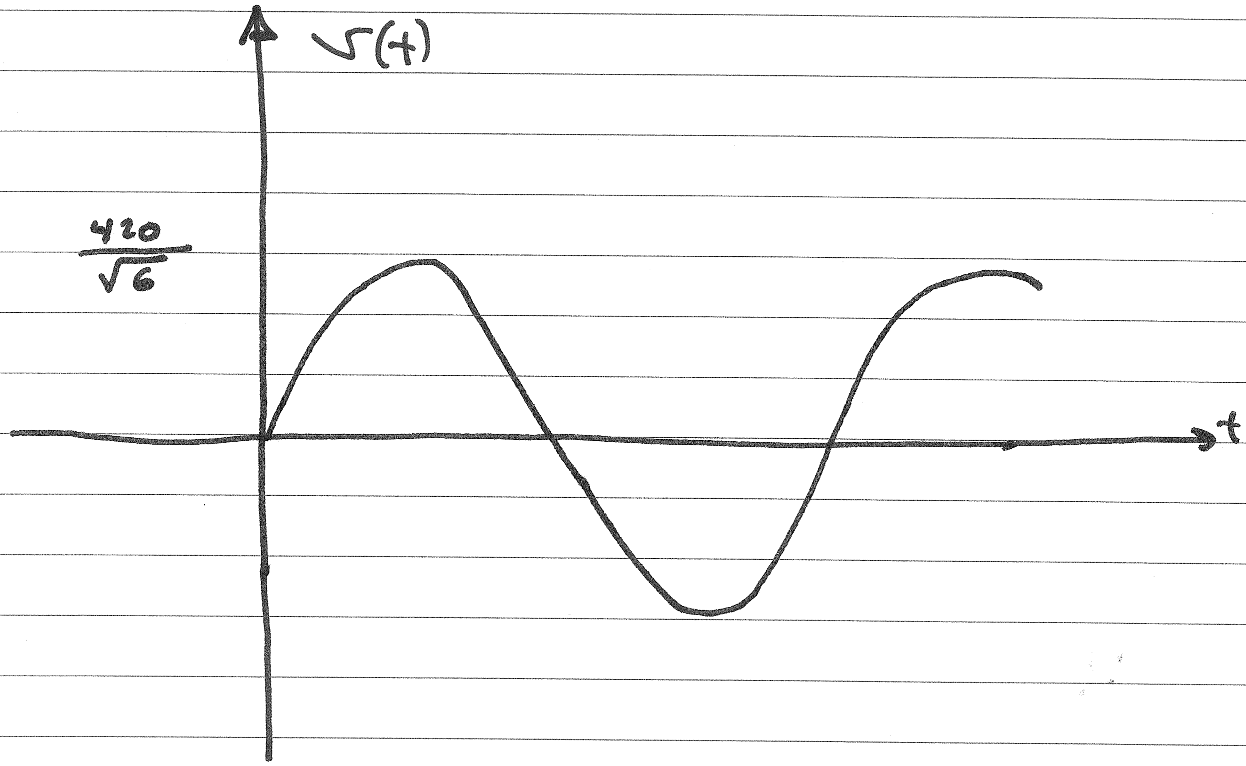
$v(0^+) = 0$, and $\frac{dv(0^+)}{dt} = 420$

$$v(0^+) = \beta_1 = 0$$

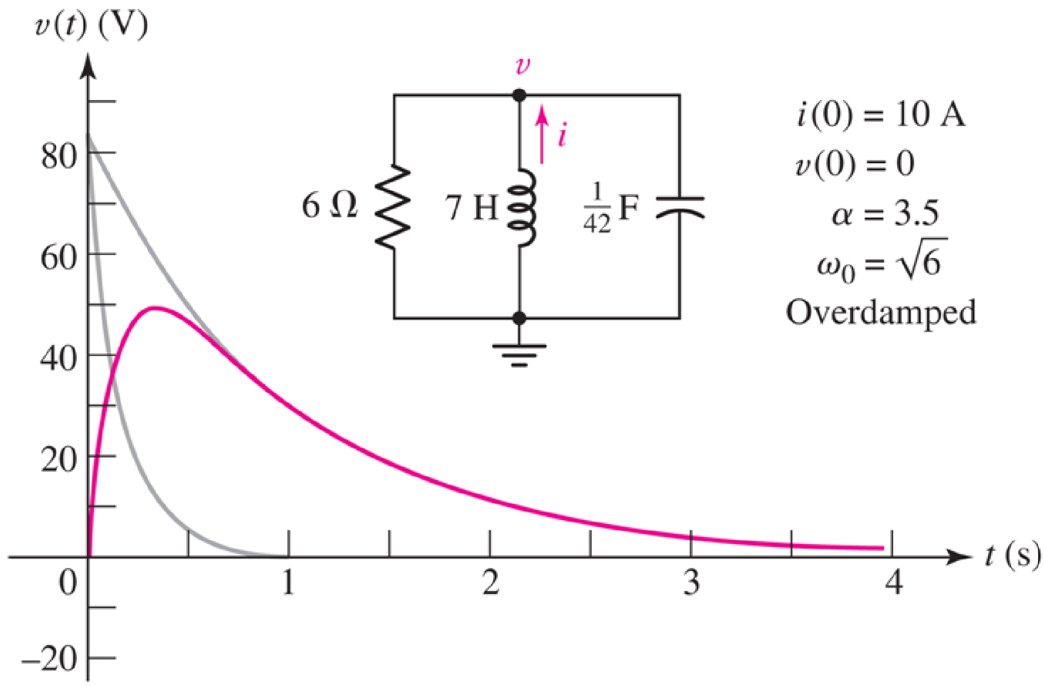
$$\frac{dv(0^+)}{dt} = \sqrt{6}\,\beta_2 = 420$$

$$\therefore \beta_2 = \frac{420}{\sqrt{6}}$$

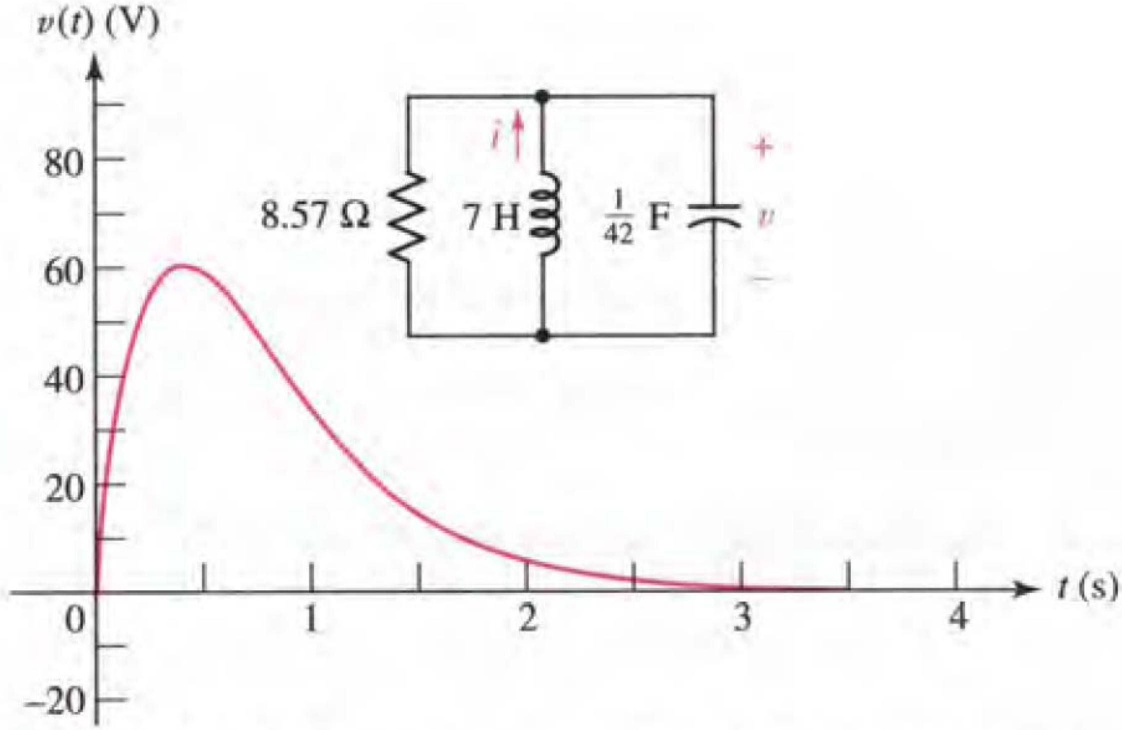
$$v(t) = \frac{420}{\sqrt{6}} \sin\sqrt{6}\, t \quad \underline{v} \quad \text{for } t > 0$$



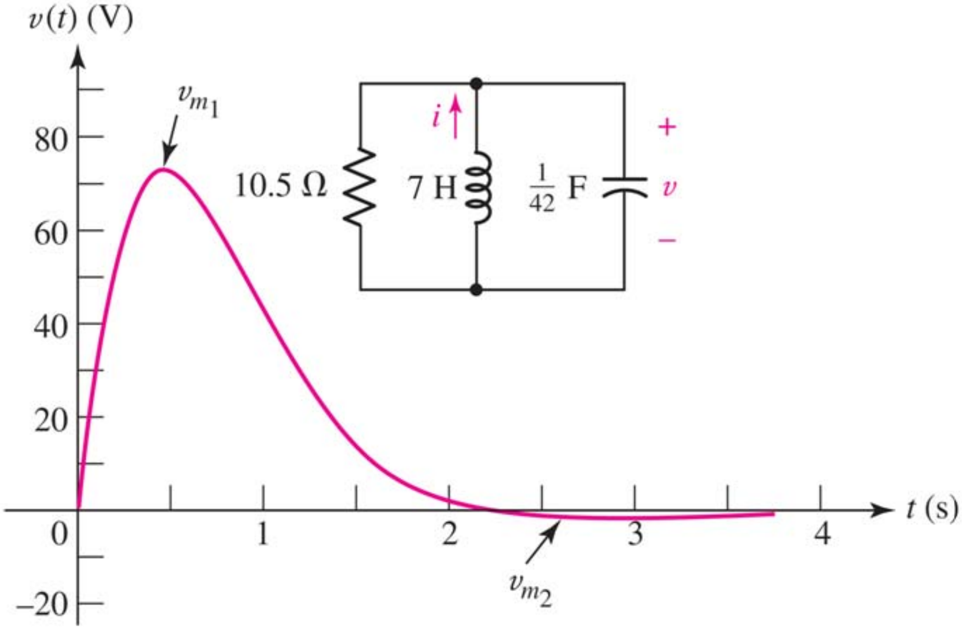
Over damped case



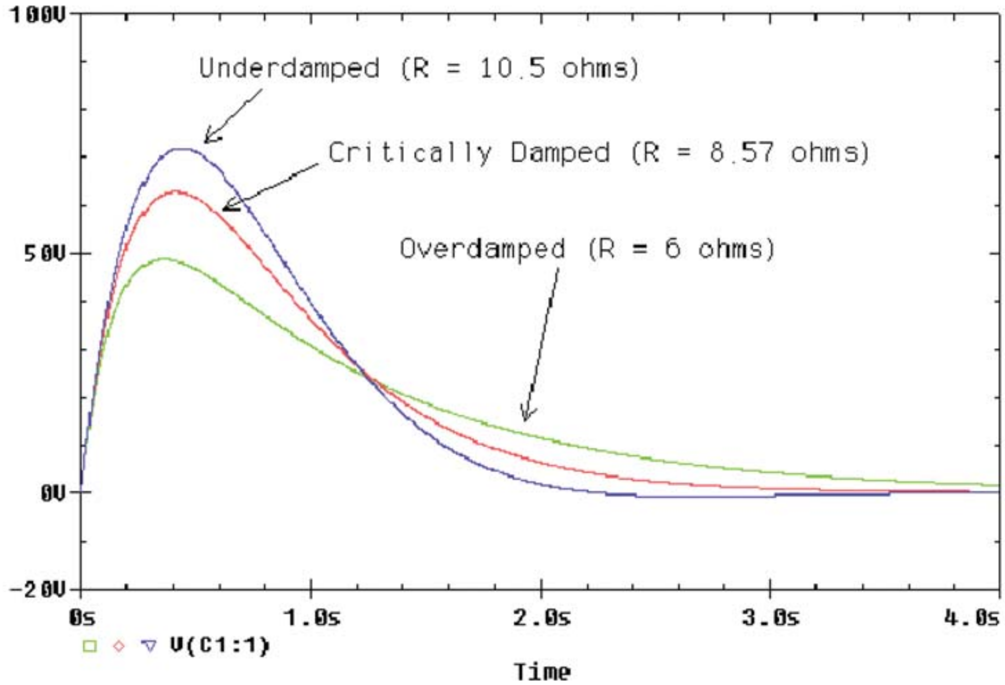
Critical damped case



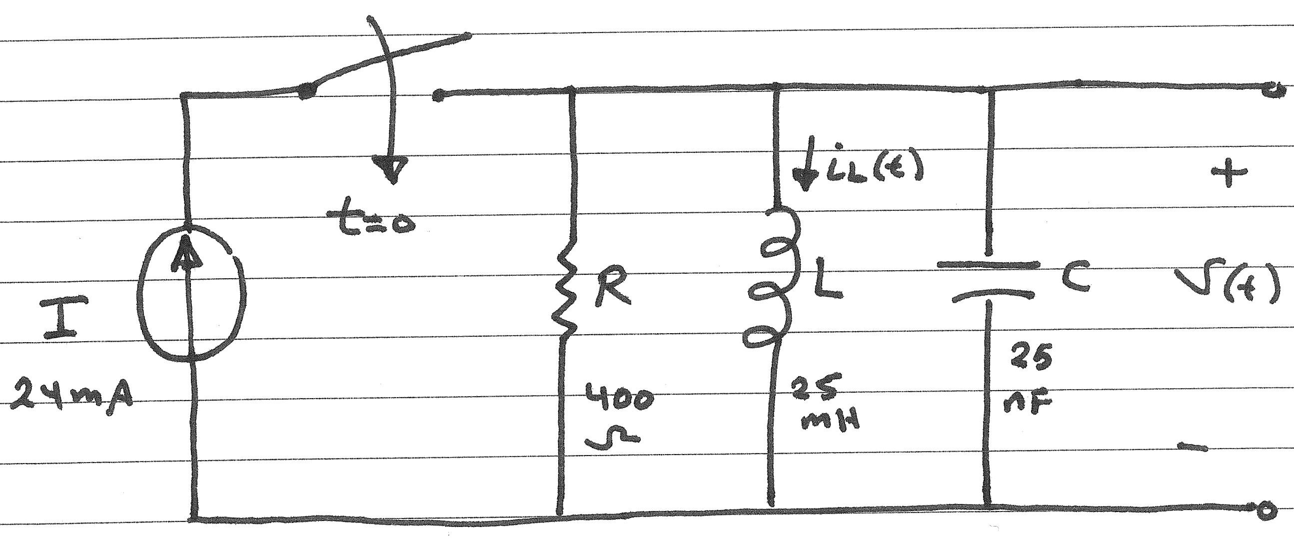
Under damped case



Comparing the Responses



Step response of Parallel RLC Circuit

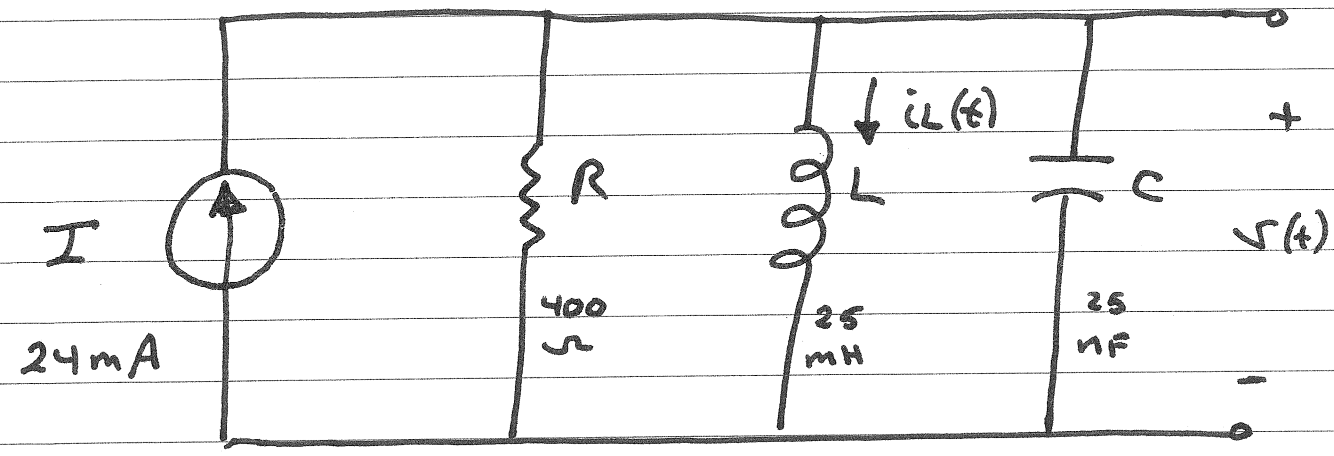


$i_L(0^-) = 0$, and $v_C(0^-) = 0$

Find $v(t)$ for $t > 0$

Find $i_L(t)$ for $t > 0$

For $t > 0$



$i_L(0^-) = 0$, and $v_C(0^-) = 0$, Find $i_L(t)$

KCL :

$$I = i_R(t) + i_L(t) + i_C(t)$$

$$I = \frac{v(t)}{R} + i_L(t) + C\frac{dv(t)}{dt}$$

$$v(t) = v_L(t) = L\frac{di_L(t)}{dt}$$

$$\therefore \quad I = LC\frac{d^2 i_L(t)}{dt^2} + \frac{L}{R}\frac{di_L(t)}{dt} + i_L(t)$$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{di_L(t)}{dt} + \frac{1}{LC}i_L(t) = \frac{I}{LC}$$

second order nonhomogeneouse diff. equation

$$i_L(t) = i_n(t) + i_f(t)$$

$i_n(t)$ = natural response

$i_f(t)$ = forced response

To find $i_f(t)$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{d i_L(t)}{dt} + \frac{1}{LC} i_L(t) = \frac{I}{LC}$$

let $i_f(t) = K$

$$0 + 0 + \frac{1}{LC} i_f(t) = \frac{I}{LC}$$

$$\therefore \quad i_f(t) = I = K$$

To find $i_n(t)$

$$\frac{d^2 i_L(t)}{dt^2} + \frac{1}{RC}\frac{d i_L(t)}{dt} + \frac{1}{LC} i_L(t) = 0$$

$$s^2 + \frac{1}{RC} s + \frac{1}{LC} = 0$$

$$S_1 = -\frac{1}{2RC} + \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$S_2 = -\frac{1}{2RC} - \sqrt{\left(\frac{1}{2RC}\right)^2 - \frac{1}{LC}}$$

$$S_1 = -20000$$

$$S_2 = -80000$$

Since S_1, S_2 are real and unequal

$\therefore$ we have overdamped case

$$\therefore \quad i_{Ln}(t) = A_1 e^{-20000t} + A_2 e^{-80000t}$$

$$\therefore \quad i_L(t) = i_{Lf}(t) + i_{Ln}(t)$$

$$i_L(t) = 24\,mA + A_1 e^{-20000t} + A_2 e^{-80000t} \qquad t>0$$

To find A_1, and A_2, we need

$i_L(0^+)$ and $\dfrac{di_L(0^+)}{dt}$

$$i_L(0^+) = i_L(0^-) = 0$$

$$v_C(t) = v_L(t) = L\frac{di_L(t)}{dt}$$

$$\therefore v_C(0^+) = L\frac{di_L(0^+)}{dt} = v_C(0^-) = 0$$

$$\therefore \frac{di_L(0^+)}{dt} = 0$$

$$i_L(t) = 24\,mA + A_1 e^{-20000t} + A_2 e^{-80000t} \quad t>0$$

$$i_L(0^+) = 24\,mA + A_1 + A_2$$

$$\therefore A_1 + A_2 = -24\,mA \quad \text{——— } (1)$$

$$\frac{di_L(0^+)}{dt} = -20000A_1 - 80000A_2 = 0 \quad \text{——— } (2)$$

Solving (1) and (2), we get

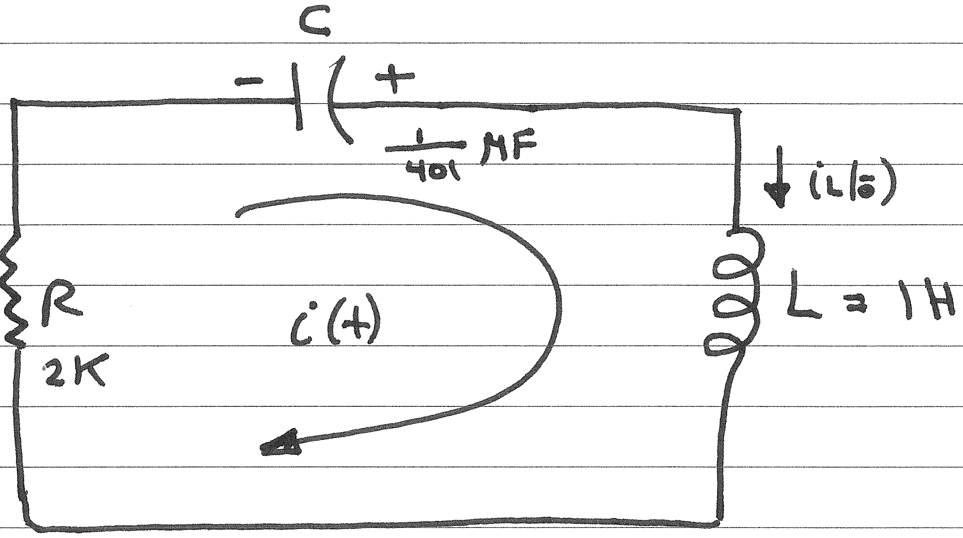
$$A_1 = -32\,mA$$

$$A_2 = 8\,mA$$

$$\therefore i_L(t) = \left(24 - 32e^{-20000t} + 8e^{-80000t}\right) mA, \text{ for } t>0$$

Natural Response of Series RLC Circuit

For $t > 0$



$v_C(0^-) = V_0$, and $i_L(0^-) = I_0$

Find $i(t)$ for $t > 0$

KVL :

$$L\frac{di(t)}{dt} + Ri(t) - v_C(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt = 0$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt = v_C(0^-) \qquad \text{— (1)}$$

Differentiation of (1)

$$L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t) = 0$$

second order homogeneouse diff. equation.

$$Ls^2 + Rs + \frac{1}{C} = 0$$

$$s_1 = -\frac{R}{2L} + \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

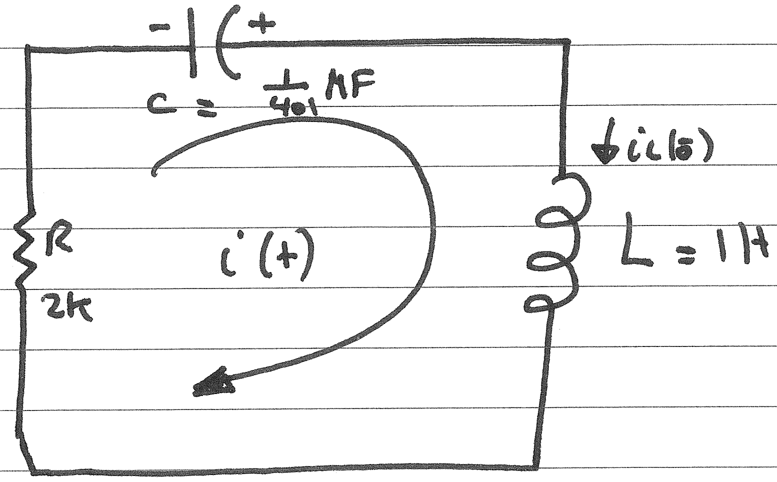
$$s_2 = -\frac{R}{2L} - \sqrt{\left(\frac{R}{2L}\right)^2 - \frac{1}{LC}}$$

Let $\alpha = \frac{R}{2L}$

$\omega_0 = \frac{1}{\sqrt{LC}}$

$$\therefore \quad s_1 = -\alpha + \sqrt{\alpha^2 - \omega_0^2}$$

$$s_2 = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$



let $v_C(0^-) = V_0 = 2V$

$i_L(0^-) = I_0 = 2mA$

$$\alpha = \frac{R}{2L} = 1000$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = 20025$$

Since $\alpha < \omega_0$

$\therefore$ we have underdamped case

$$\therefore \omega_d = \sqrt{\omega_0^2 - \alpha^2} = 20000$$

$$\therefore i(t) = e^{-\alpha t}\left(\beta_1 \cos \omega_d t + \beta_2 \sin \omega_d t\right) \quad \text{for } t > 0$$

$$\therefore i(t) = e^{-1000t}\left(\beta_1 \cos 20{,}000t + \beta_2 \sin 20{,}000t\right) \quad \text{for } t > 0$$

To find B_1 and B_2, we need

to have $i(0^+)$ and $\frac{di(0^+)}{dt}$

$$i(0^+) = i_L(0^+) = i_L(0^-) = 2mA$$

$$L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{0^-}^{t} i(t)dt - v_C(0^-) = 0$$

at $t = 0^+$

$$L\frac{di(0^+)}{dt} + Ri(0^+) + 0 - v_C(0^-) = 0$$

$$\therefore \frac{di(0^+)}{dt} = \frac{v_C(0^-) - Ri(0^+)}{L} = -2$$

$$i(t) = e^{-1000t}\left(\beta_1 \cos 20000t + \beta_2 \sin 20000t\right)$$

$$i(0^+) = \beta_1 = 2mA$$

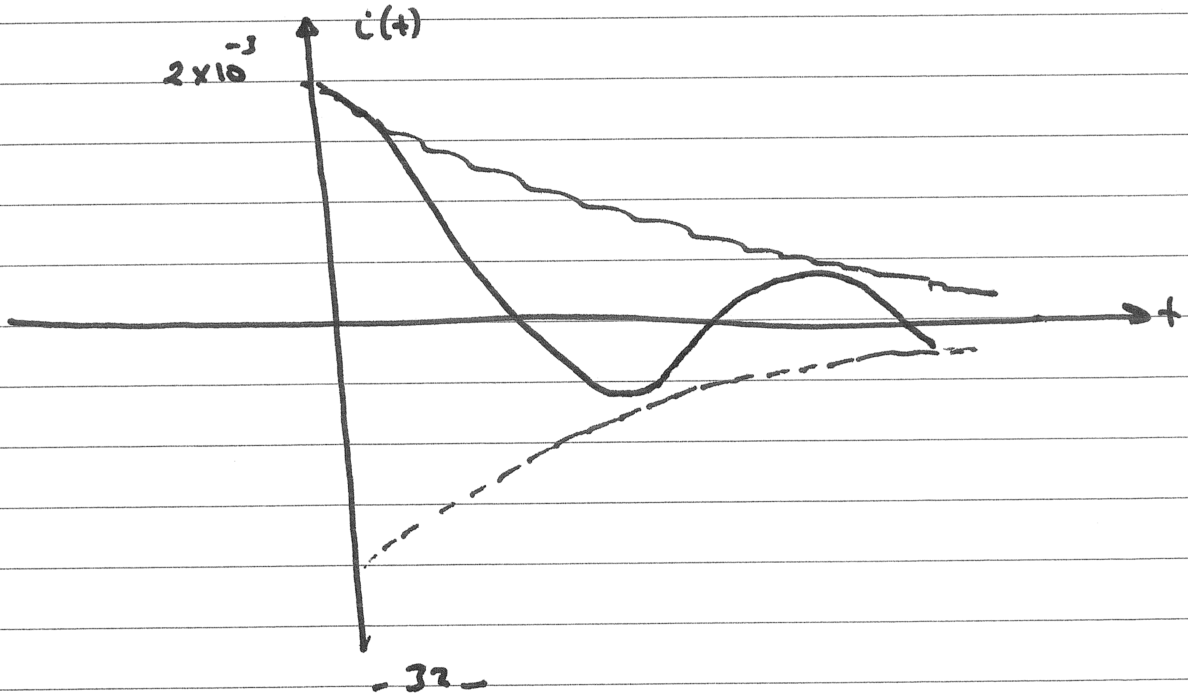
$$\therefore \quad \beta_1 = 2\ mA$$

$$\frac{di(0^+)}{dt} = 20000\,\beta_2 - 2\times10^{-3}(1000)$$

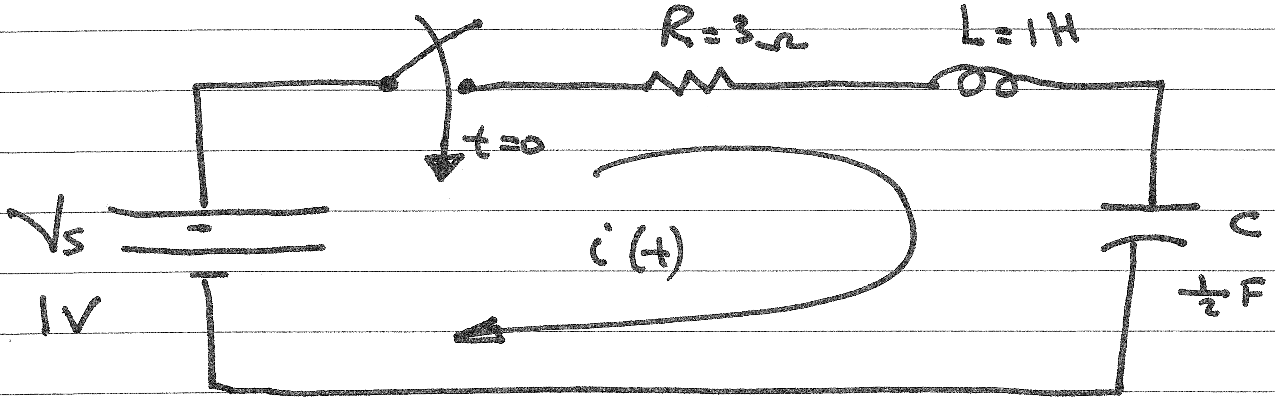
$$\frac{di(0^+)}{dt} = 20000 - 2 = -2$$

$$\therefore \quad \beta_2 = 0$$

$$\therefore \quad i(t) = 2\,e^{-1000t}\cos 20000t \quad mA, \quad \text{for } t>0$$



Step response of series RLC Circuit



$v_c(0^-) = 0 \quad , \quad i_L(0^-) = 0$

Find $i(t)$ for $t > 0$

KVL :

$$V_s = R\,i(t) + L\frac{di(t)}{dt} + v_c(0^-) + \frac{1}{c}\int_{0^-}^{t} i(t)\,dt$$

$$0 = L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{c}\,i(t)$$

$\therefore \; i(t) = i_n(t)$

$$0 = LS^2 + RS + \frac{1}{C}$$

$$0 = S^2 + 3S + 2$$

$\therefore \; S_1 = -1 \quad , \quad S_2 = -2$

S_1, S_2 are real and unequal

$\therefore$ overdamped Case

$$\therefore \; i(t) = A_1 e^{-t} + A_2 e^{-2t} \quad \text{for } t > 0$$

or

$$\alpha = \frac{R}{2L} = 1.5$$

$$\omega_0 = \frac{1}{\sqrt{LC}} = \sqrt{2}$$

$\alpha > \omega_0 \longrightarrow$ overdamped Case

To find A_1 and A_2

$$i(0^+) = i(0^-) = 0$$

$$V_s = R i(t) + L \frac{di(t)}{dt} + \frac{1}{C} \int_{0^-}^{t} i(t) + V_c(0^-)$$

<u>at $t = 0^+$</u>

$$V_s = R i(0^+) + L \frac{d i(0^+)}{dt} + 0 + 0$$

$$\therefore \; \frac{di(0^+)}{dt} = \frac{V_s - R i(0^+)}{L} = \frac{V_s}{L} = \frac{1}{1} = 1$$

$$i(0^+) = 0$$

$$\frac{di(0^+)}{dt} = 1$$

$$i(t) = A_1 e^{-t} + A_2 e^{-2t} \quad \text{for } t>0$$

$$i(0^+) = A_1 + A_2 = 0 \quad \text{———} \quad (1)$$

$$\frac{di(0^+)}{dt} = -A_1 - 2A_2 = 1 \quad \text{———} \quad (2)$$

Solving (1) and (2)

$$A_1 = 1 \quad , \quad A_2 = -1$$

$$\therefore i(t) = \left(e^{-t} - e^{-2t}\right) \underline{\underline{A}} \quad , \text{ for } t>0$$

$$v_c(t) = v_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt$$

$$v_c(t) = \left(1 - 2e^{-t} + e^{-2t}\right) \underline{\underline{V}}$$

for $t>0$

To find $v_c(t)$

$$V_s = Ri(t) + L\frac{di(t)}{dt} + v_c(t)$$

$$i(t) = i_c(t) = C\frac{dv_c(t)}{dt}$$

$$V_s = RC\frac{dv_c(t)}{dt} + LC\frac{d^2v_c(t)}{dt^2} + v_c(t)$$

second order nonhomogeneouse diff. equation

$$\therefore\ v_c(t) = v_{cn}(t) + v_{cf}(t)$$

$$v_{cf}(t) = K$$

$$K = V_s$$

$$\therefore\ v_c(t) = V_s + v_{cn}(t)$$

$$0 = LCs^2 + RCs + 1$$

$$0 = \frac{1}{2}s^2 + \frac{3}{2}s + 1$$

$$\therefore\ S_1 = -1$$

$$S_2 = -2$$

$$v_c(t) = v_{cn}(t) + v_{cf}(t)$$

$$v_c(t) = A_1 e^{-t} + A_2 e^{-2t} + 1 \qquad \text{for } t > 0$$

To find A_1, A_2

$$v_c(0^+) = v_c(0^-) = 0$$

$$i(t) = i_L(t) = i_C(t) = C \frac{dv_c(t)}{dt}$$

$$\therefore \quad i_L(0^+) = i_C(0^+) = C \frac{dv_c(0^+)}{dt} = 0$$

$$\therefore \quad \frac{dv_c(0^+)}{dt} = 0$$

$$\therefore \quad v_c(0^+) = 0$$

$$\frac{dv_c(0^+)}{dt} = 0$$

$$v_c(t) = 1 + A_1 e^{-t} + A_2 e^{-2t}$$

$$v_c(0^+) = 1 + A_1 + A_2 = 0$$

$$\therefore \quad A_1 + A_2 = -1 \qquad \text{——— } (1)$$

$$\frac{dv_c(t)}{dt} = -A_1 e^{-t} - 2A_2 e^{-2t}$$

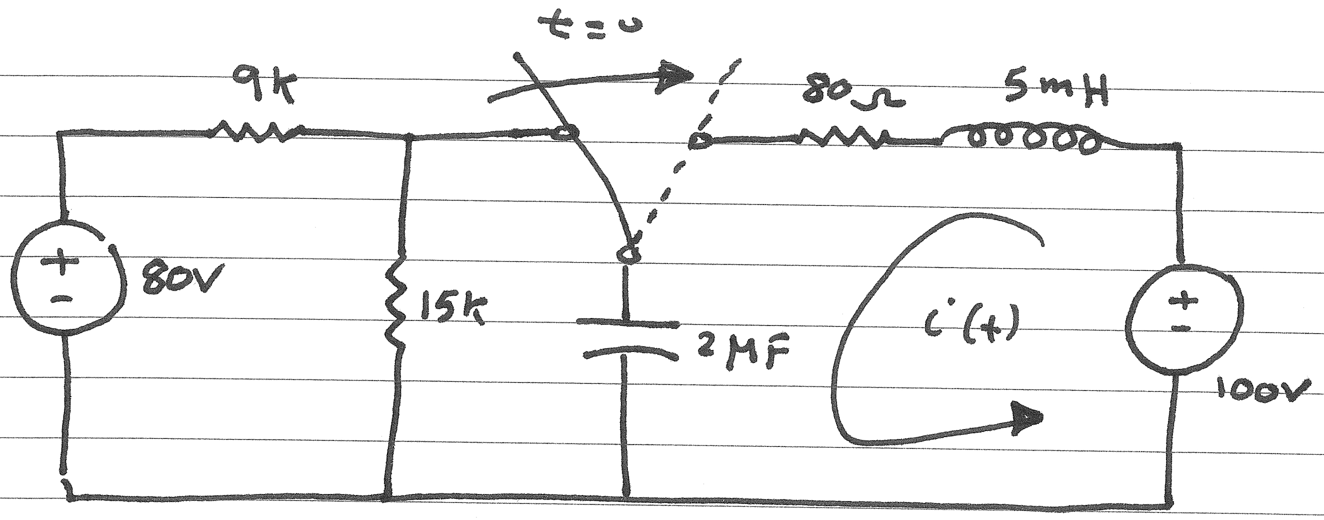
$$\frac{dv_c(0^+)}{dt} = -A_1 - 2A_2 = 0 \qquad \text{——— } (2)$$

Solving ① and ②

$A_1 = -2$

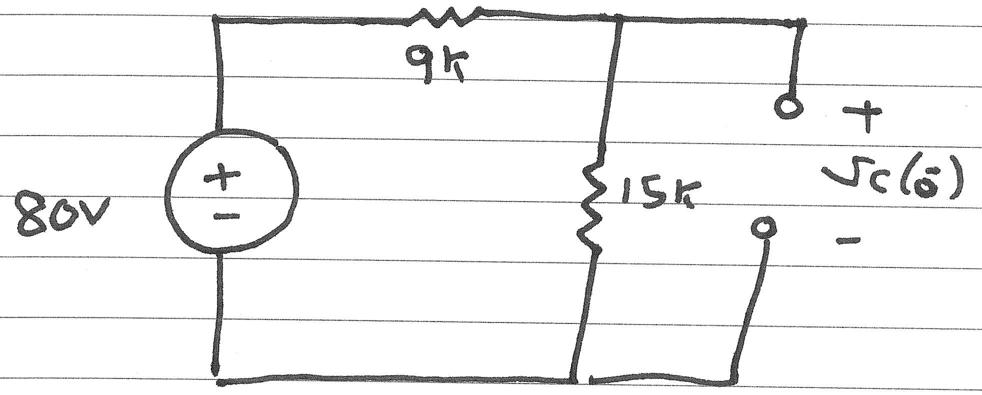
$A_2 = 1$

$\therefore V_c(t) = \left(1 - 2e^{-t} + e^{-2t}\right) \underline{\underline{V}}$ for $t>0$



1) Find $i(t)$ for $t > 0$

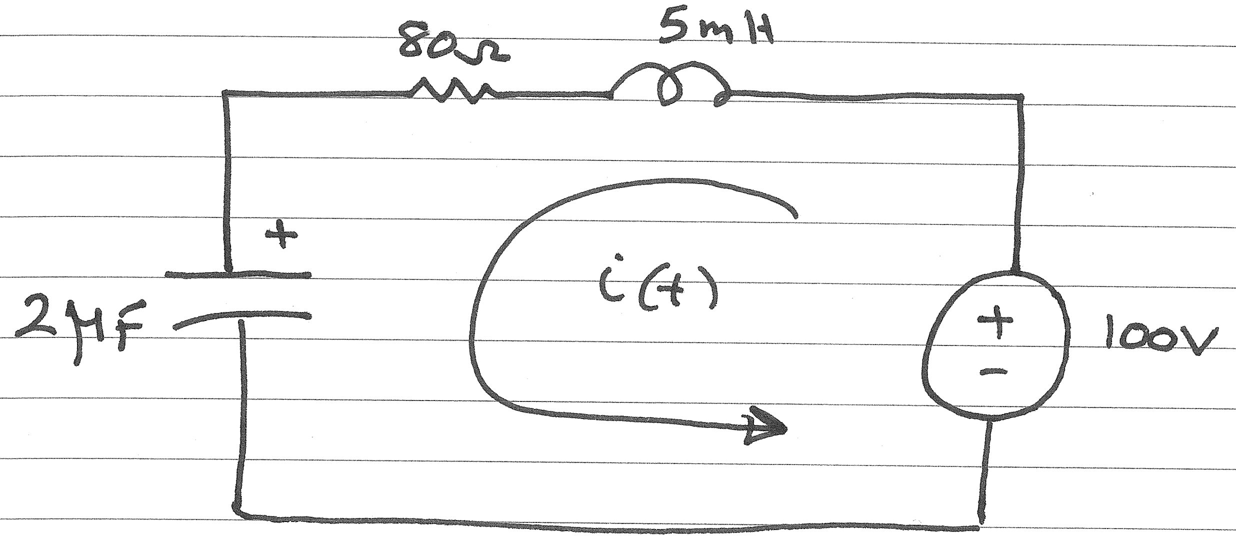
a) For $t < 0$, $t = 0^-$



$$V_c(0^-) = \frac{15k}{15k + 9k} \, 80 = 50V$$

$$i_L(0^-) = 0$$

b) For $t > 0$



KVL :

$$100 = L\frac{di(t)}{dt} + Ri(t) + v_c(0^-) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt$$

$$50 = L\frac{di(t)}{dt} + Ri(t) + \frac{1}{C}\int_{0^-}^{t} i(t)\,dt$$

$\frac{d}{dt}$:

$$0 = L\frac{d^2 i(t)}{dt^2} + R\frac{di(t)}{dt} + \frac{1}{C}i(t)$$

second order homogeneous diff. equation

$\therefore\ i(t) = i_n(t)$

Let $v_1(t) = 10 \sin(5t - 30^\circ)$

$v_2(t) = 15 \sin(5t + 10^\circ)$

$\therefore$ $v_2(t)$ Leads $v_1(t)$ by 40°

Let $i_1(t) = 2 \sin(377t + 45^\circ)$

$i_2(t) = 0.5 \cos(377t + 10^\circ)$

$\cos \alpha = \sin(\alpha + 90^\circ)$

$0.5 \cos(377t + 10^\circ) = 0.5 \sin(377t + 100^\circ)$

$\therefore$ $i_2(t)$ leads $i_1(t)$ by 55°

$$0 = LCs^2 + RCs + 1$$

$$0 = 10 \times 10^{-9} s^2 + 160 \times 10^{-6} s + 1$$

$\therefore \quad s_1 = -8000 + j\,6000$

$\quad\;\; s_2 = -8000 - j\,6000$

underdamped Case

$$i(t) = e^{-8000t}\left(\beta_1 \cos 6000t + \beta_2 \sin 6000t\right)$$

To find β_1 and β_2

$$i(0^+) = i(0^-) = 0$$

$$\frac{di(0^+)}{dt} = \frac{V_s - V_c(0^-)}{L} = 10{,}000$$

$$\beta_1 = 0$$

$$\beta_2 = 1.67$$

$$i(t) = 1.67\,e^{-8000t} \sin 6000t \; \underline{A} \quad \text{for } t > 0$$

2) Find $v_c(t)$ for $t > 0$

$$V_s = L\frac{di(t)}{dt} + Ri(t) + v_c(t)$$

$$i(t) = i_c(t) = C\frac{dv_c(t)}{dt}$$

$$\therefore \quad V_s = LC\frac{d^2v_c(t)}{dt^2} + RC\frac{dv_c(t)}{dt} + v_c(t)$$

second order nonhomogeneouse diff. equation

$$\therefore \quad v_c(t) = v_{cn}(t) + v_{cf}(t)$$

$$v_{cf}(t) = K$$

$$V_s = 0 + 0 + K$$

$$\therefore \quad v_{cf} = K$$

To find $v_{cn}(t)$

$$0 = LCs^2 + RCs + 1$$

$$0 = 10\times10^{-9}s^2 + 160\times10^{-6}s + 1$$

$$s_1 = -8000 + j\,6000$$
$$s_2 = -8000 - j\,6000$$

Underdamped Case

$$v_c(t) = 100 + e^{-8000t}\left(\beta_1 \cos 6000t + \beta_2 \sin 6000t\right)$$

To find β_1, and β_2, we need

$v_c(0^+)$ and $\frac{dv_c(0^+)}{dt}$

$$v_c(0^+) = v_c(0^-) = 50\,V$$

$$i_c(0^+) = i_L(0^+) = \frac{C\,dv_c(0^+)}{dt} = 0$$

$$\therefore \frac{dv_c(0^+)}{dt} = 0$$

$$v_c(0^+) = 100 + \beta_1 = 50$$

$$\therefore \beta_1 = -50$$

$$\frac{dv_c(0^+)}{dt} = -8000\beta_1 + 6000\beta_2$$

$$\therefore \beta_2 = -66.67$$

$$\therefore v_c(t) = \left[100 + e^{-8000t}\left(-50\cos 6000t - 66.67 \sin 6000t\right)\right] V$$

for $t > 0$

Network Analysis II
ENEE 335

3 Credit Hours
Second Semester 2014

Instructor : Dr. Atheer Barghouthi
Date : 24.02.2014

Intended Learning Outcomes (ILO’s)

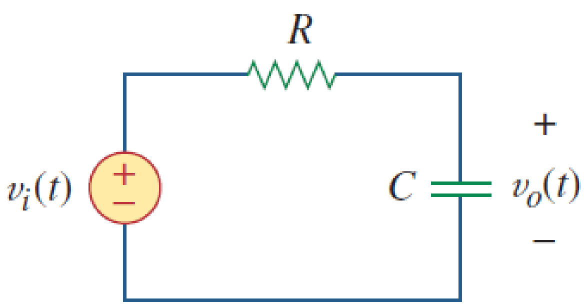
- To be able to apply the linear network analysis methods in the Laplace domain, (mesh analysis, node analysis, and network theorems and circuits transformation).
- To be able to apply the circuit synthesis methods in the implementation of LTI systems (transfer functions).
- To understand two ports elements representation.
- To be able to solve circuits with two ports elements
- To be able to determine and analyze the frequency response of the systems
- To be able to analyze different types of analog filters (active and passive).
- To be able to design and implement different types of analog filters
- To understand the graph representation of electric networks
- To apply the graph theory concepts in solving electric networks
- To be able to use CAD tools (ORCAD , MATLAB) in simulating and synthesizing electric networks
- To acquire interaction and communication skills

Filters Design

- **Passive filters**

Low pass filters: $H(s)=\dfrac{K}{(s/\alpha)+1}$

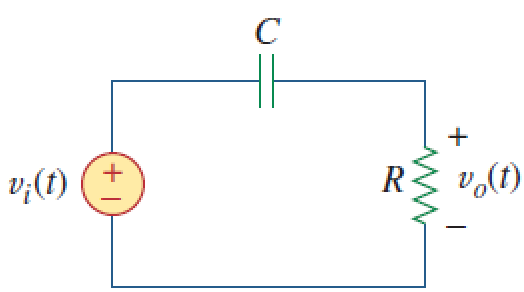
$$H(s)=\frac{1}{sRC+1}$$



Gain = 1
Cut off frequency =1/RC

High pass filters: $H(s)=\dfrac{K(s/\alpha)}{(s/\alpha)+1}$

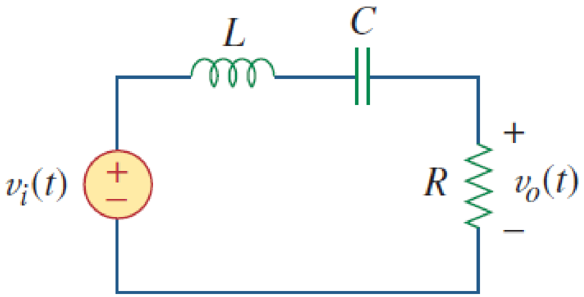
$$H(s)=\frac{(s/\alpha)}{(s/\alpha)+1}$$



Gain = 1
Cut off frequency =1/RC

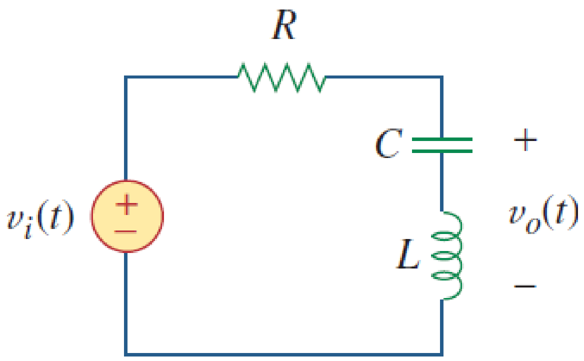
Filters Design

- **Passive Bandpass filters: Second order filters RLC, K =1**



$$T(s) = \frac{K(s/\omega_0)}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$

- **Passive Bandstop filters: Second order filters RLC, K=1**



$$T(s) = \frac{K[(s/\omega_0)^2 + 1]}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$

Filters Design

- **Why active filters?**

- They provide frequency selectivity comparable to passive *RLC* circuits plus passband gains greater than one.
- They have OP AMP outputs, which means that the chain rule applies in a cascade design.
- They do not require inductors, which can be large, lossy, and expensive in low-frequency applications.

- **Filter design**

- Filter Design begins with the specifications, which is translated into the order of the filter and the corresponding transfer function

- The filter is implemented depending on its order using first and second order building blocks(2nd order complex poles are used to get steeper slopes, less components, and lower cost)

- The building blocks components can be scaled to reach components that can be implemented. Additionally, they can be frequency scaled to shift cut off frequencies.

Filters Design

- **Transfer functions general form of second order filters**

Low-pass filters have the following form

$$T(s) = \frac{K}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$

High-pass filters have the following form

$$T(s) = \frac{K(s/\omega_0)^2}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$

Band-pass filters have the following form

$$T(s) = \frac{K(s/\omega_0)}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$

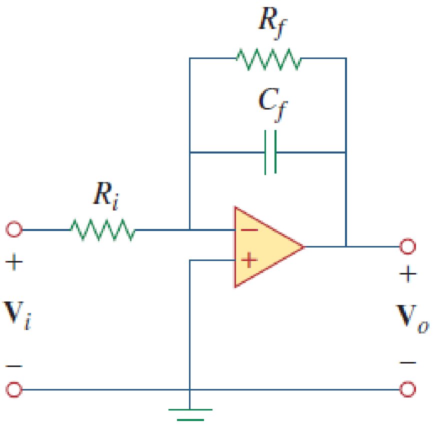
Finally, band-stop filters have the following form

$$T(s) = \frac{K[(s/\omega_0)^2 + 1]}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$

Filters Design

- **First order active filters building blocks**

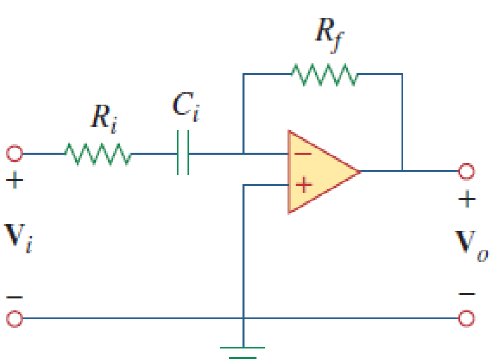
Low pass filters: $H(s) = \dfrac{K}{(s/\alpha)+1}$



$$H(s) = -\frac{R_f}{R_i}\frac{1}{sR_fC_i+1}$$

Gain = -Rf/Ri
Cut off frequency =1/RfCf

High pass filters: $H(s) = \dfrac{K(s/\alpha)}{(s/\alpha)+1}$



$$H(s) = -\frac{R_f}{R_i}\frac{sR_fC_i}{sR_fC_i+1}$$

Gain = -Rf/Ri
Cut off frequency =1/RfCi

Filters Design

- **Second order active filters building blocks**

$$T(s) = \frac{\text{numerator}}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1} = \frac{\text{numerator}}{(s/\omega_0)^2 + \frac{B}{\omega_0}(s/\omega_0) + 1} = \frac{\text{numerator}}{(s/\omega_0)^2 + \frac{1}{Q}(s/\omega_0) + 1}$$

- **Low pass filters and high pass filters: gain, cut off frequency and zeta**
- **Band pass and band stop filters: bandwidth, Q**

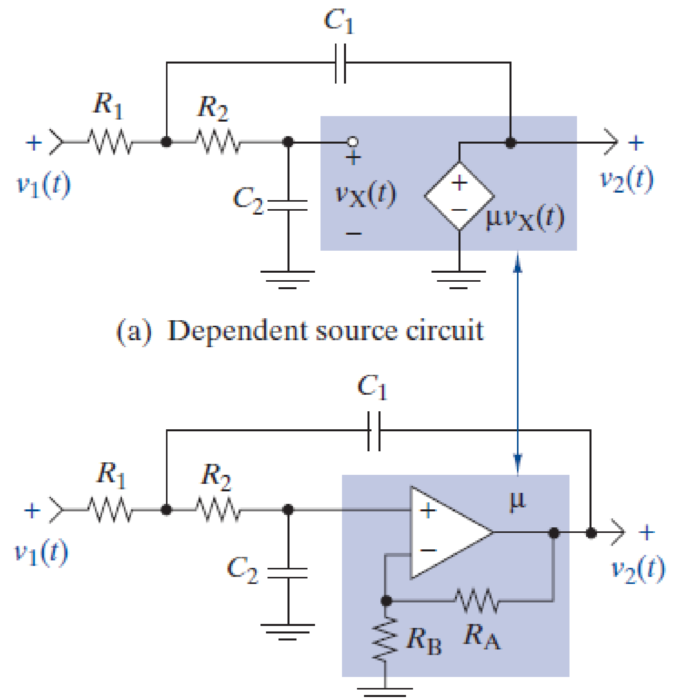
Filters Design

- **Second order active filters building blocks**

Low pass filters: $T(s) = \dfrac{K}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{\mu}{R_1R_2C_1C_2\,s^2 + (R_1C_1 + R_1C_2 + R_2C_2 - \mu R_1C_1)s + 1}$$

$$\sqrt{R_1R_2C_1C_2} = 1/\omega_0 \quad \text{and} \quad R_1C_2 + R_2C_2 + (1-\mu)R_1C_1 = 2\zeta/\omega_0$$



(a) Dependent source circuit

(b) OP AMP realization

1) Equal element method:

The *equal element* method requires that $R_1 = R_2 = R$ and $C_1 = C_2 = C$.

$$RC = \frac{1}{\omega_0} \quad \text{and} \quad \mu = 3 - 2\zeta$$

Using this method, we select values of R (or C) and R_B, then solve for C (or R) and $R_A = (\mu - 1)R_B$. The dc gain achieved by this method is $|T(0)| = \mu = 3 - 2\zeta$, and the method is valid for $\zeta < 1.5$.

Filters Design

- **Second order active filters building blocks**

2) Unity gain method: $R_1 = R_2 = R$ and $\mu = 1$

$$R\sqrt{C_1 C_2} = \frac{1}{\omega_0} \quad \text{and} \quad \frac{C_2}{C_1} = \zeta^2$$

Using this method, we select a value of C_1 and calculate $C_2 = \zeta^2 C_1$ and $R = (\omega_0\sqrt{C_1 C_2})^{-1}$. To get a gain $\mu = 1$, we make the noninverting OP AMP circuit a voltage follower. That is, we replace R_A by a short circuit and R_B by an open circuit. This eliminates the need for R_A and R_B but requires two different capacitors. Obviously the dc gain achieved by this design method is $|T(0)| = \mu = 1$. The unity gain method does not place any restrictions on the value of the damping coefficient, ζ.

The equal element and unity gain methods provide alternative ways to design an active low-pass filter with prescribed values of ω_0 and ζ. However, the dc gains produced by these methods are predetermined and are not adjustable design parameters. An additional gain correction stage may be needed when ω_0, ζ, and the dc gain are all three prescribed.

Filters Design

- **Second order active filters building blocks**

Example:

Develop a second-order low-pass transfer function with a corner frequency at $\omega_0 = 1$ krad/s and with corner frequency gain equal to the dc gain. Use MATLAB to help visualize the Bode plots of the desired transfer function. Then design two competing circuits using the equal element and unity gain design techniques. Use OrCAD to simulate the frequency responses of the designed circuits. Compare the results and comment on any differences.

Low pass: $T(s) = \dfrac{K}{(s/1000)^2 + 2\zeta(s/1000) + 1}$

The requirement that the corner frequency gain equal the dc gain results in $|T(j1000)| = K/2\zeta$. These two gains are equal when $\zeta = 0.5$.

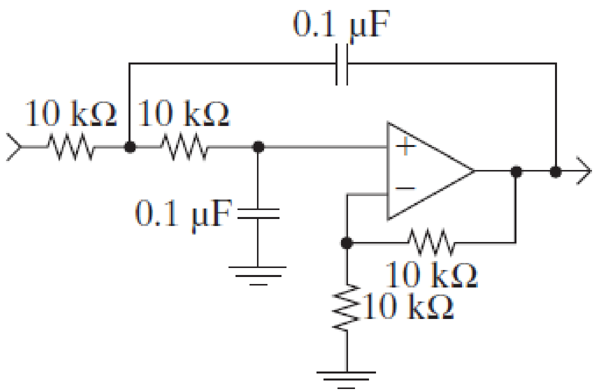
Equal element design: $RC = 10^{-3}$ s and $\mu = 2$

$$T_1(s) = \frac{2 \times 10^6}{s^2 + 1000s + 10^6}$$

Selecting $R = R_B = 10\,\text{k}\Omega$ requires $C = 0.1\,\mu\text{F}$ and $R_A = (\mu - 1)R_B = 10\,\text{k}\Omega$.

Filters Design

- **Second order active filters building blocks**

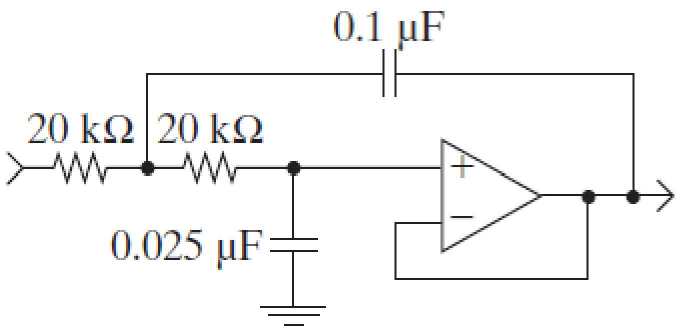


$$\textit{Unity gain design:}\quad T_1(s) = \frac{10^6}{s^2 + 1000s + 10^6}$$

With the recognition that $R\sqrt{C_1C_2} = 10^{-3}$ s and $C_2 = \zeta^2 C_1 = 0.25\,C_1$, selecting $C_1 = 0.1\,\mu\text{F}$ dictates that $C_2 = 0.025\,\mu\text{F}$ and $R = 20\,\text{k}\Omega$. The dc gain in this circuit is, by design, 1 and Figure 14–4(b) shows the resulting circuit.

Filters Design

- **Second order active filters building blocks**



Ex2:

Develop a second-order low-pass transfer function with a corner frequency of 50 rad/s, a dc gain of 2, and a gain of 4 at the corner frequency. Validate your result by using MATLAB to plot the transfer function's absolute gain versus frequency.

Answer: The desired transfer function is

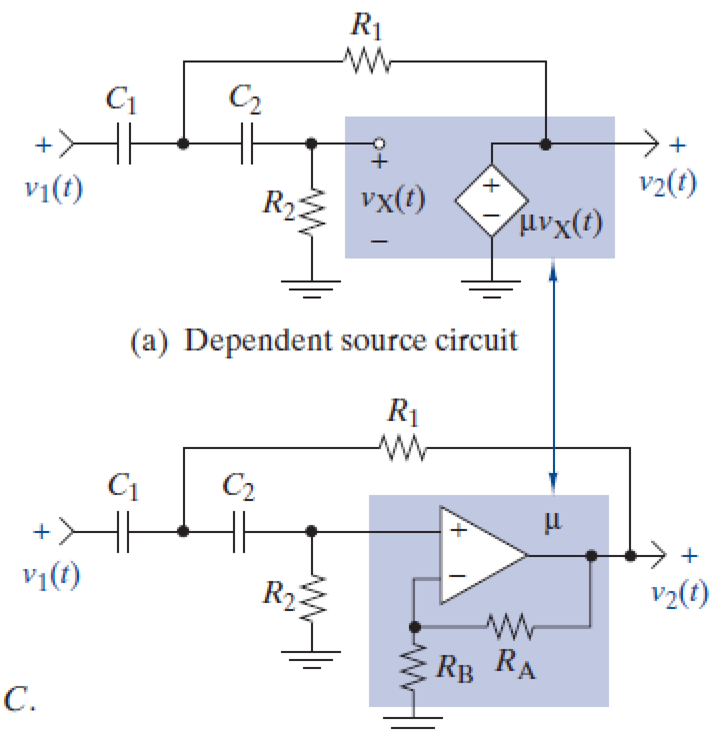
$$T(s) = \frac{5000}{s^2 + 25s + 2500}$$

Filters Design

- **Second order active filters building blocks**

High pass filters: $T(s) = \dfrac{K(s/\omega_0)^2}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$

High pass filters can be achieved from low pass filters with the following transformation: replacing s/ω_0 by ω_0/s



(a) Dependent source circuit

(b) OP AMP realization

$$\sqrt{R_1R_2C_1C_2} = \frac{1}{\omega_0} \quad \text{and} \quad \sqrt{\frac{R_1C_1}{R_2C_2}} + \sqrt{\frac{R_1C_2}{R_2C_1}} + (1-\mu)\sqrt{\frac{R_2C_2}{R_1C_1}} = 2\zeta$$

The *equal element method* requires that $R_1 = R_2 = R$ and $C_1 = C_2 = C$.

$$RC = \frac{1}{\omega_0} \quad \text{and} \quad \mu = 3 - 2\zeta$$

Under this method we select values of R (or C) and R_B, and solve for C (or R) and $R_A = (\mu - 1)R_B$. The infinite-frequency gain achieved by this method is $|T(\infty)| = \mu = 3 - 2\zeta$, and the method is valid for $\zeta < 1.5$.

Filters Design

- **Second order active filters building blocks**

High pass filters: $T(s) = \dfrac{K(s/\omega_0)^2}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$

The *unity gain method* requires that $C_1 = C_2 = C$ and $\mu = 1$.

$$C\sqrt{R_1 R_2} = \frac{1}{\omega_0} \quad \text{and} \quad \frac{R_1}{R_2} = \zeta^2$$

Using this method, we select a value of R_2 and calculate $R_1 = \zeta^2 R_2$ and $C = (\omega_0\sqrt{R_1R_2})^{-1}$. To get a gain $\mu = 1$, we make the noninverting OP AMP circuit into a voltage follower. That is, we replace R_A by a short circuit and R_B by an open circuit, thereby eliminating the need for these two resistors. Obviously the infinite-frequency gain achieved by this method is $|T(\infty)| = \mu = 1$. As with the low-pass filter, the unity gain method does not place any restrictions on the value of the damping coefficient, ζ.

The equal element and unity gain methods are alternative ways to design an active high-pass filter with prescribed values of ω_0 and ζ. As we found in the low-pass case, the passband gains produced by these methods are predetermined and are not adjustable design parameters. An additional gain correction stage may be needed when ω_0, ζ, and the infinite-frequency gain are all three prescribed.

Filters Design

- **Second order active filters building blocks**

High pass filters example:

Develop a second-order high-pass transfer function with a corner frequency at $\omega_0 = 20$ krad/s, an infinite-frequency gain of 0 dB, and a corner frequency gain of −3 dB. Use MATLAB to help visualize the Bode magnitude plot of the desired transfer function. Then design two competing circuits using the equal element and unity gain design techniques. Use OrCAD to simulate the frequency responses of the designed circuits. Compare the results and comment on the differences.

SOLUTION:

The required transfer function has the form

$$T(s) = \frac{K(s/20{,}000)}{(s/20{,}000)^2 + 2\zeta(s/20{,}000) + 1}$$

The infinite-frequency gain is $T(\infty) = K$ and the corner frequency gain is $|T(j20,000)| = K/2\zeta$. A gain of 0 dB at infinite frequency requires $K = 1$. A gain of −3 dB at the corner frequency requires $|T(j20{,}000)| = 1/\sqrt{2}$, which in turn requires that $\zeta = 1/\sqrt{2} = 0.707$. Therefore, the desired transfer function is

$$T(s) = \frac{s^2}{s^2 + 28{,}280s + 400 \times 10^6}$$

Filters Design

- **Second order active filters building blocks**

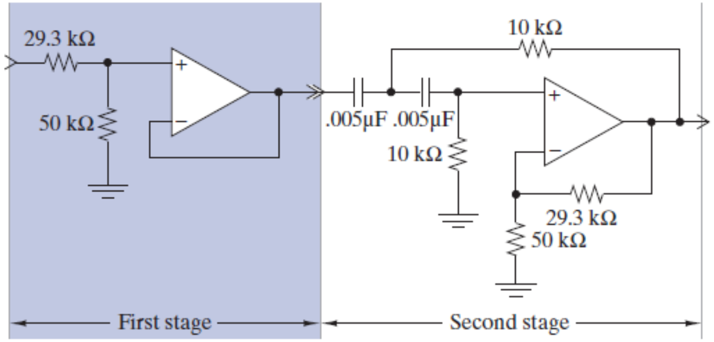
High pass filters example:

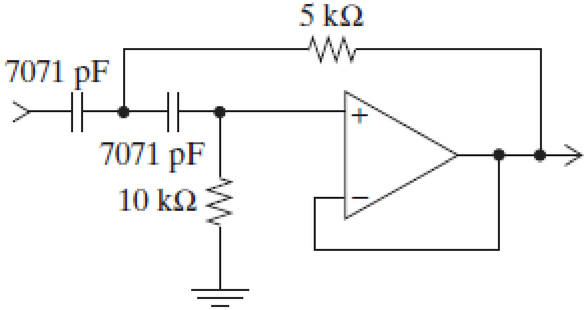
Equal element design: Inserting $\omega_0 = 20$ krad/s and $\zeta = 0.707$ into Eq. (14–13) yields $RC = 5 \times 10^{-5}$ s and $\mu = 3 - \sqrt{2} = 1.586$. Selecting $C = 0.005\,\mu$F and $R_B = 50\,k\Omega$ requires $R = 10\,k\Omega$ and $R_A = (\mu - 1)R_B = 29.3\,k\Omega$. The high-frequency gain of this design is $|T(\infty)| = \mu = 1.586$, which is more than the specified value of 1 (0 dB). We add a gain correction stage with a gain of $1/1.586 = 0.6305$ to bring the overall gain down to 0 dB. Figure 14–11(a) shows the resulting two-stage design.

Unity gain design: Inserting $\omega_0 = 20$ krad/s and $\zeta = 0.707$ into Eq. (14–14) yields $C\sqrt{R_1R_2} = 5 \times 10^{-5}$ s and $R_1 = \zeta^2 R_2 = 0.5R_2$. Selecting $R_2 = 10\,k\Omega$ requires that $R_1 = 5\,k\Omega$ and $C = 7071$ pF. The $|T(\infty)|$ gain of this circuit is 1, which matches the desired 0 dB. The resulting single-stage design is shown in Figure 14–11(b).

The circuits were created in OrCAD and simulated using AC Sweep. Their Probe results are shown in Figure 14–12. Clearly, both circuit designs implement the transfer function extremely well and meet all three design specifications.

Comparing the two circuit designs shows that the *equal element design* requires an extra OP AMP used as a buffer and four more resistors than the *unity gain design* because its gain is greater than 1. The voltage divider could be placed after the filter thereby eliminating the need for the buffer, but raising a concern about





Filters Design

- **Second order active filters building blocks**

High pass filters example 2:

Construct a second-order high-pass transfer function with a corner frequency of 20 rad/s, an infinite-frequency gain of 4, and a gain of 2 at the corner frequency.

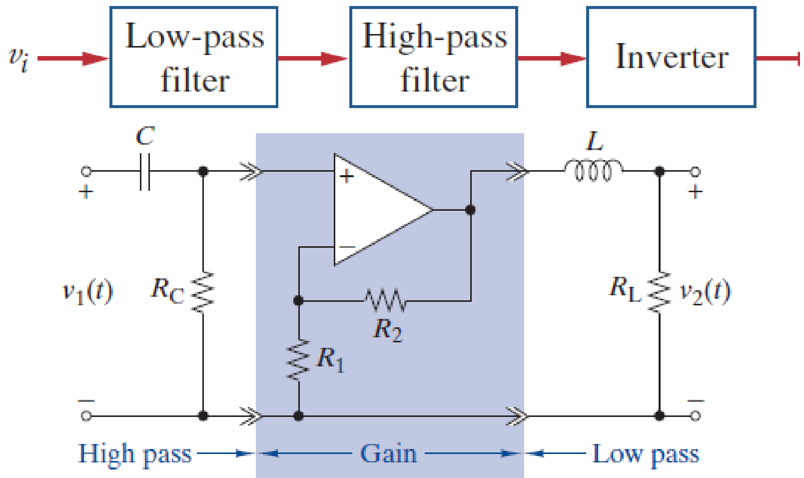
Answer:
$$T(s) = \frac{4s^2}{s^2 + 40s + 400}$$

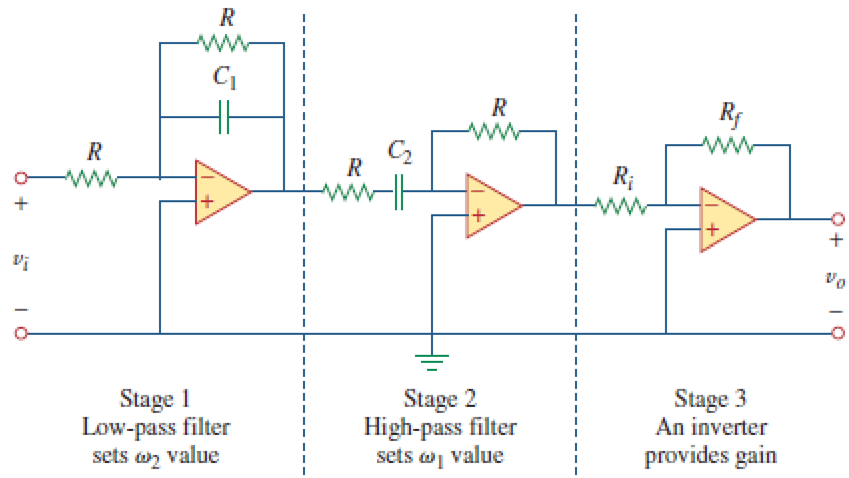
	Equal Element	Unity Gain
Low Pass	$R_1 = R_2 = R$ $C_1 = C_2 = C$ $\omega_0 = {}^1/_{RC}$ $\mu = 3 - 2\zeta$	$R_1 = R_2 = R$ $\mu = 1$ $\omega_0 = 1/R\sqrt{C_1 C_2}$ $\zeta = \sqrt{C_2/C_1}$
High Pass	$R_1 = R_2 = \mathrm{R}$ $C_1 = C_2 = C$ $\omega_0 = {}^1/_{RC}$ $\mu = 3 - 2\zeta$	$C_1 = C_2 = C$ $\mu = 1$ $\omega_0 = 1/C\sqrt{R_1 R_2}$ $\zeta = \sqrt{R_1/R_2}$

Filters Design

- **Active Bandpass filters: Two approaches to design band pass filters**

1. Cascade of a low pass and high pass filter (used for low Q filters or wideband)





$$T(s) = \frac{K(s/\omega_0)}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$

$$T(s) = T_1(s) \times T_2(s) = \underbrace{\left(\frac{K_1 s}{s+\alpha_1}\right)}_{\text{high pass}} \underbrace{\left(\frac{K_2}{s+\alpha_2}\right)}_{\text{low pass}}$$

$$\omega_0 = \sqrt{\omega_1 \omega_2}$$

$$B = \omega_2 - \omega_1$$

$$Q = \frac{\omega_0}{B}$$

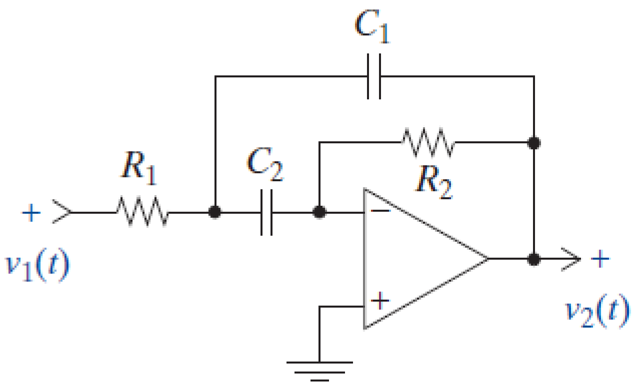
$$K = \frac{R_f}{R_i} \frac{\omega_2}{\omega_1 + \omega_2}$$

Filters Design

- **Active Bandpass filters:**

2. Direct design (used for narrow band filters or high Q)

$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{-R_2C_2s}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_1C_2)s + 1}$$



$$\sqrt{R_1R_2C_1C_2} = 1/\omega_0 \quad \text{and} \quad R_1C_1 + R_1C_2 = 2\zeta/\omega_0$$

$$\sqrt{R_1R_2C_1C_2} = \frac{1}{\omega_0} \quad \text{and} \quad \sqrt{\frac{R_1C_1}{R_2C_2}} + \sqrt{\frac{R_1C_2}{R_2C_1}} = 2\zeta$$

$$\textit{equal-capacitor method}(C_1 = C_2 = C)$$

$$\sqrt{R_1R_2}C = \frac{1}{\omega_0} \quad \text{and} \quad \frac{R_1}{R_2} = \zeta^2$$

Under this method we select a value of R_2 and solve for $R_1 = \zeta^2 R_2$ and $C = (\omega_0\sqrt{R_1R_2})^{-1}$. Since this method uses $C_1 = C_2$, the center frequency gain found from Eq. (14–16) is $|T(j\omega_0)| = R_2/2R_1 = 1/2\zeta^2$. Note that the center frequency gain is greater than 1 when $\zeta < 1/\sqrt{2}$. For example, when $\zeta = 0.1$, the gain is 50. This contrasts with the passive *RLC* bandpass circuit, whose center frequency gain is always 1.

Filters Design

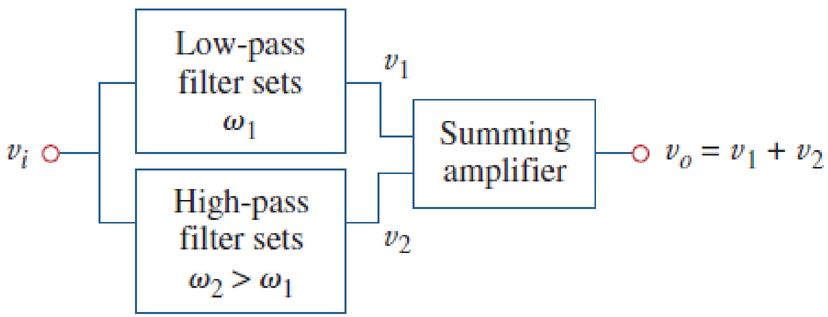
Bandpass active filters:

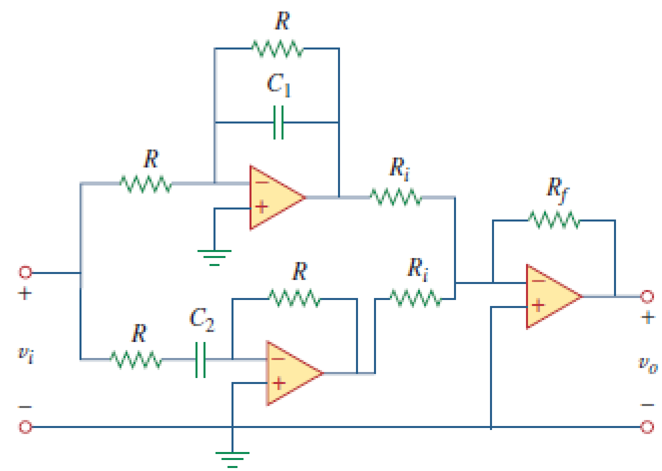
The key descriptive parameters of a second-order bandpass filter are its **center frequency** ω_0 and **bandwidth** $B = 2\zeta\omega_0$. It is customary to add a third parameter called the **quality factor**, defined as $Q = \omega_0/B$. From this definition it is clear that Q and ζ are both dimensionless parameters related as $Q = 1/2\zeta$. Either parameter can be used to characterize filter bandwidth. When $Q > 1$ ($\zeta < 0.5$), the filter is said to be narrow-band because the bandwidth is less than the center frequency. When $Q < 1$ ($\zeta > 0.5$), the filter is said to be wide-band. The active bandpass building block developed here is best suited to narrow-band applications. Filters with a high Q are also referred to as **tuned** filters.

Filters Design

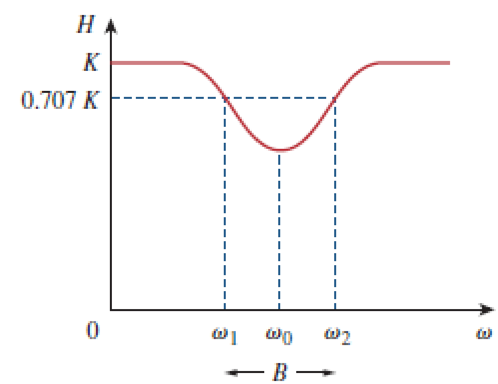
Bandstop active filters:

1. Parallel combination of a low pass and high pass filter





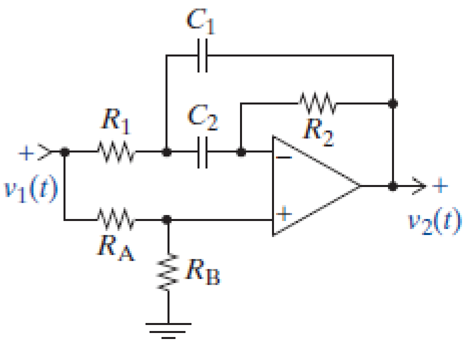
$$T(s) = \frac{K[(s/\omega_0)^2 + 1]}{(s/\omega_0)^2 + 2\zeta(s/\omega_0) + 1}$$



Filters Design

Bandstop active filters:

2. Direct design (used for narrow band filters or high Q)



$$T(s) = \frac{V_2(s)}{V_1(s)} = \frac{R_B}{R_A + R_B}\left[\frac{R_1R_2C_1C_2s^2 + (R_1C_1 + R_1C_2 - R_2C_2R_A/R_B)s + 1}{R_1R_2C_1C_2s^2 + (R_1C_1 + R_1C_2)s + 1}\right]$$

$$\sqrt{R_1R_2C_1C_2} = \frac{1}{\omega_0} \quad \text{and} \quad \sqrt{\frac{R_1C_1}{R_2C_2}} + \sqrt{\frac{R_1C_2}{R_2C_1}} = 2\zeta$$

For the circuit to perform as band stop $R_1(C_1 + C_2) - R_2C_2R_A/R_B = 0$

The *equal-capacitor method* $(C_1 = C_2 = C)$

$$\sqrt{R_1R_2}C = \frac{1}{\omega_0} \quad \text{and} \quad \frac{R_1}{R_2} = \zeta^2$$

$$\frac{R_A}{R_B} = \frac{2R_1}{R_2} \qquad \text{let } R_A = 2R_1 \text{ and } R_B = R_2.$$

Second-order bandstop filters are customarily described in terms of the **notch frequency** ω_0 and the **notch bandwidth** $B = 2\zeta\omega_0$.

Filters Design

Scaling: magnitude scaling and frequency scaling

1. **Magnitude scaling is what we used to achieve a realizable design from a prototype.**

Magnitude scaling is the process of increasing all impedances in a network by a factor, the frequency response remaining unchanged.

$$R' = K_m R, \qquad L' = K_m L$$
$$C' = \frac{C}{K_m}, \qquad \omega' = \omega$$

2. Frequency scaling is used to change the cut off frequency of the filter .

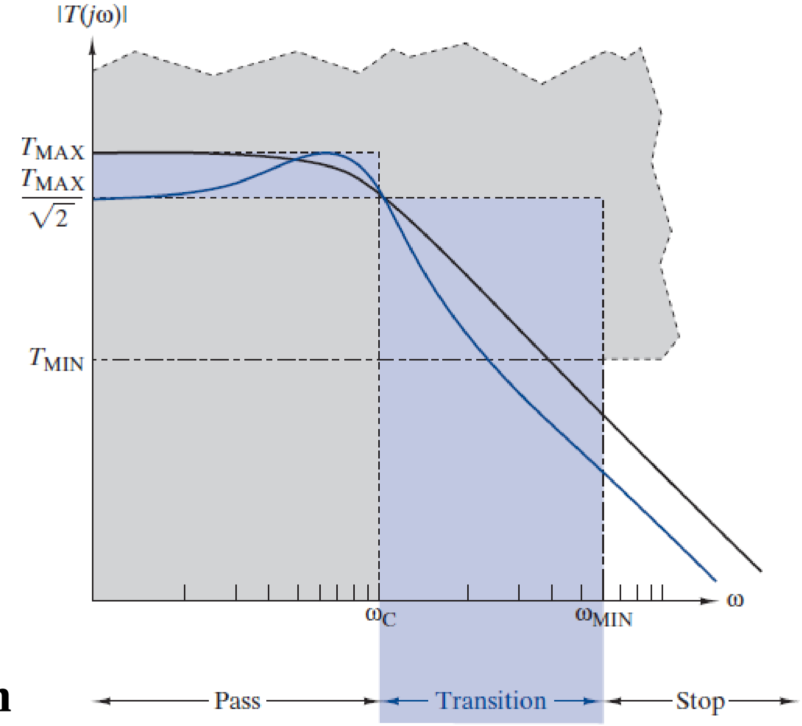
Frequency scaling is the process of shifting the frequency response of a network up or down the frequency axis while leaving the impedance the same.

$$R' = R, \qquad L' = \frac{L}{K_f}$$
$$C' = \frac{C}{K_f}, \qquad \omega' = K_f \omega$$

Filters Design

Filter design from specifications:

1. **Determine which type of the filter to design (First order cascade, Butterworth...)**
2. **Determine which type of the filter to design (low pass, high pass, band pass, band stop)**
3. **The specifications are, the cut off frequency(frequencies), the gain, and a frequency Wmin at which the gain should be lower the Tmin**
4. **Determine by a formula the order of the filter (number of stages)**
5. **Find the transfer function of each stage, from table**
6. **Frequency scale the transfer functions to fit with the cut off frequency specified**
7. **Build and connect the transfer functions to finish the design.**



Filters Design

Filter design from specifications:

Many types of filters are available, we will learn only (First order cascade, Butterworth...) the others Chebychev, Bessel... are not considereed here

Different filter typesgenerate different filter orders for a certain specifications, additionally, the quality of their magnitude and phase responses differ.

The core design is low pass filter, because it can be converted to the other three by simple tricks, it will be considered in details for the first order cascade and Butterworth

Filters Design

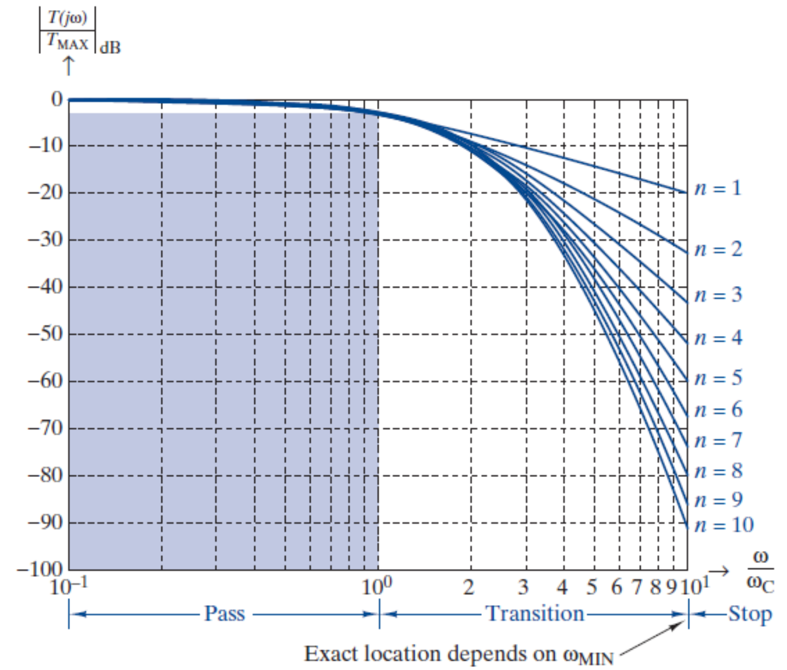
Filter design from specifications:

1. First order cascade low pass

$$T(s) = \underbrace{\left[\frac{K}{s/\alpha + 1}\right] \times \left[\frac{K}{s/\alpha + 1}\right] \times \cdots \times \left[\frac{K}{s/\alpha + 1}\right]}_{n \text{ stages}} = \frac{K^n}{(s/\alpha + 1)^n}$$

$$|T(j\omega)| = \frac{|K|^n}{\left[\sqrt{1 + (\omega/\alpha)^2}\right]^n}$$

$$\alpha = \frac{\omega_C}{\sqrt{2^{1/n} - 1}} \qquad K = (T_{MAX})^{1/n}$$



Filters Design

Filter design from specifications:
1. First order cascade low pass example

(a) Construct a first-order cascade transfer function that meets the following requirements: $T_{MAX} = 10\,\text{dB}, \omega_C = 200\,\text{rad/s}, T_{MIN} = -10\,\text{dB}$, and $\omega_{MIN} = 800\,\text{rad/s}$. Use MATLAB to visualize the gain plot.

(b) Design a cascade of active *RC* circuits that realizes the transfer function developed in (a). Use OrCAD to simulate the expected frequency response. Compare your result with the MATLAB plot.

SOLUTION:

(a) The specification requires the gain to decrease by 20 dB in a transition band with $\omega_{MIN}/\omega_C = 4$. Figure 14–23 shows that at $\omega/\omega_C = 4$, the normalized gain is about -17 dB for $n = 2$ and about -22 dB for $n = 3$. Thus, $n = 3$ is the smallest integer that meets the transition band requirement. Given n and ω_C, we calculate α using Eq. (14–29):

$$\alpha = \frac{\omega_C}{\sqrt{2^{1/n} - 1}} = \frac{200}{\sqrt{2^{1/3} - 1}} = 392\,\text{rad/s}$$

Since $T_{MAX} = 10\,\text{dB}$ (factor of $\sqrt{10}$), we write $K = (\sqrt{10})^{1/3} = 1.468$. So, finally, the required first-order cascade transfer function is

$$T(s) = \left(\frac{1.468}{s/392 + 1}\right)^3$$

Note that the cutoff frequency of each stage ($\alpha = 392\,\text{rad/s}$) is greater than the cutoff frequency of the n-stage transfer function ($\omega_C = 200\,\text{rad/s}$). A quick look at Eq. (14–29) reveals that $\alpha > \omega_C$ for all $n > 1$.

Filters Design

Filter design from specifications:

2. Butterworth low pass

$$|T(j\omega)| = \frac{|K|}{\sqrt{1 + (\omega/\omega_C)^{2n}}} \qquad n \geq \frac{1}{2}\frac{\ln\left[(T_{MAX}/T_{MIN})^2 - 1\right]}{\ln\left[\omega_{MIN}/\omega_C\right]}$$

For example, if the transition band gain must decrease by 30 dB ($T_{MAX}/\ T_{MIN} = 10^{3/2}$) in the two octaves above cutoff ($\omega_{MIN}/\omega_C = 4$), then Eq. (14–31) yields

$$n \geq \frac{1}{2}\frac{\ln\left[(10^{3/2})^2 - 1\right]}{\ln\left[4\right]} = 2.49$$

Order	Normalized Denominator Polynomials
1	$(s+1)$
2	$(s^2 + 1.414s + 1)$
3	$(s+1)(s^2+s+1)$
4	$(s^2 + 0.7654s + 1)(s^2 + 1.848s + 1)$
5	$(s+1)(s^2 + 0.6180s + 1)(s^2 + 1.618s + 1)$
6	$(s^2 + 0.5176s + 1)(s^2 + 1.414s + 1)(s^2 + 1.932s + 1)$

Network Analysis II
ENEE 335

3 Credit Hours
Second Semester 2014

Instructor : Dr. Atheer Barghouthi
Date : 24.02.2014

Intended Learning Outcomes (ILO's)

- To be able to apply the linear network analysis methods in the Laplace domain, (mesh analysis, node analysis, and network theorems and circuits transformation).
- To be able to apply the circuit synthesis methods in the implementation of LTI systems (transfer functions).
- To understand two ports elements representation.
- To be able to solve circuits with two ports elements
- To be able to determine and analyze the frequency response of the systems
- To be able to analyze different types of analog filters (active and passive).
- To be able to design and implement different types of analog filters
- To understand the graph representation of electric networks
- To apply the graph theory concepts in solving electric networks
- To be able to use CAD tools (ORCAD , MATLAB) in simulating and synthesizing electric networks
- To acquire interaction and communication skills

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Laplace transform

- **Laplace transform motivation**

As we have learned, we can change the circuit from time domain to phasor(frequency domain) to make the circuit easier to analyse using sinusoidal signals. The draw back of this approach was that we were only able to analyse for the steady state. The transient analysis could not be obtained using phasor domain analysis. Additionally int time domain solving differential equations of higher orders is not an easy task to do. Transforming the circuit into laplace domain enables us to have a general solution of the circuit to any signal and for both the transient and steady state solution. In the laplace domain the equations generated are algebraic which is easier to deal with than differential equations´when the solution is obtained in the laplace domain it is inverted back to the time domain to find the final solution. Additionally, in laplace domain we are not limited to transients due to dc inputs only, but also more complicated input signals can be dealt with. Each component has current or voltage rating that should not be exceeded. Steady state can tell us that the component is operating within the limits, at the time the limit might be exceeded in transient. That is why transients are necessary to be analyzed. Additionally transients show the speed at which the circuit reaches the steady state which is important in many applications. One additional advantage, the laplace domain provide the total solution in one step and not two (forced and natural responses)

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Laplace transform

- **Laplace transform review**

The **Laplace transform** is an integral transformation of a function $f(t)$ from the time domain into the complex frequency domain, giving $F(s)$.

$$\boxed{\mathcal{L}[f(t)] = F(s) = \int_{0^-}^{\infty} f(t)e^{-st}\,dt} \qquad s = \sigma + j\omega$$

This is a unilateral laplace transform which we will consider. Bilateral laplace where the lower integration limit is minus infinity is of no interest for our systems as we care only about causal systems.

The inverse laplace transform is of complex nature, as a result we will not use it to calculate inverses but we will use look up tables.

$$\mathcal{L}^{-1}[F(s)] = f(t) = \frac{1}{2\pi j}\int_{\sigma_1 - j\infty}^{\sigma_1 + j\infty} F(s)e^{st}\,ds$$

Laplace transform

- **Laplace transform review**

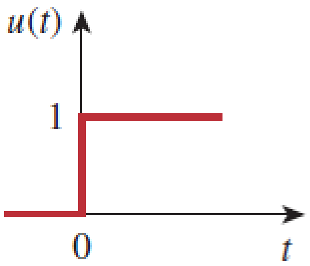
To guruantee that what we use is unilateral, the function is normally multiplied by step function.

Determine the Laplace transform of each of the following functions: (a) $u(t)$, (b) $e^{-at}u(t)$, $a \geq 0$, and (c) $\delta(t)$.

Solution:

(a) For the unit step function $u(t)$, shown in Fig. 15.2(a), the Laplace transform is

$$\mathcal{L}[u(t)] = \int_{0^-}^{\infty} 1e^{-st}\,dt = -\frac{1}{s}e^{-st}\Big|_0^{\infty} = -\frac{1}{s}(0) + \frac{1}{s}(1) = \frac{1}{s} \quad \textbf{(15.1.1)}$$



(b) For the exponential function, shown in Fig. 15.2(b), the Laplace transform is

$$\mathcal{L}[e^{-at}u(t)] = \int_{0^-}^{\infty} e^{-at}e^{-st}\,dt = -\frac{1}{s+a}e^{-(s+a)t}\Big|_0^{\infty} = \frac{1}{s+a} \quad \textbf{(15.1.2)}$$

Laplace transform

- **Laplace transform review**

Determine the Laplace transform of $f(t) = \sin \omega t\, u(t)$.

Solution:
Using Eq. (B.27) in addition to Eq. (15.1), we obtain the Laplace transform of the sine function as

$$F(s) = \mathcal{L}[\sin \omega t] = \int_0^\infty (\sin \omega t) e^{-st}\, dt = \int_0^\infty \left(\frac{e^{j\omega t} - e^{-j\omega t}}{2j} \right) e^{-st}\, dt$$

$$= \frac{1}{2j} \int_0^\infty \left(e^{-(s-j\omega)t} - e^{-(s+j\omega)t} \right) dt$$

$$= \frac{1}{2j} \left(\frac{1}{s - j\omega} - \frac{1}{s + j\omega} \right) = \frac{\omega}{s^2 + \omega^2}$$

Find the Laplace transform of $f(t) = 50 \cos \omega t\, u(t)$.

Answer: $50s/(s^2 + \omega^2)$.

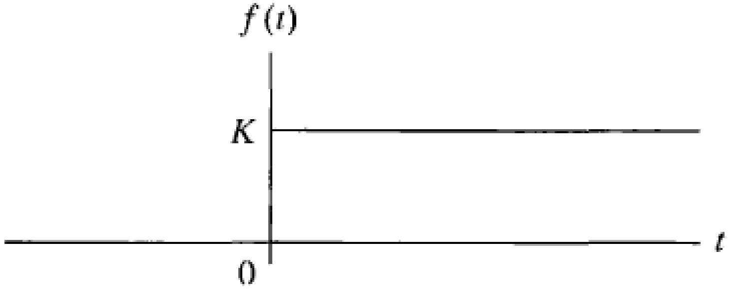
Laplace transform

- **Singularity functions**

We may encounter functions that have a discontinuity, or jump, at the origin. For example, we know from earlier discussions of transient behavior that switching operations create abrupt changes in currents and voltages. We accommodate these discontinuities mathematically by introducing the step and impulse functions.

Step function

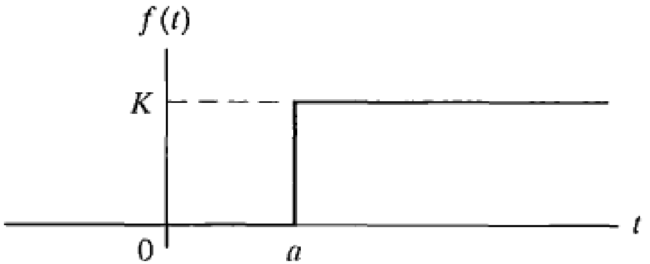
$$Ku(t) = 0, \quad t < 0,$$
$$Ku(t) = K, \quad t > 0.$$



The step function is not defined at $t = 0$. In situations where we need to define the transition between 0^- and 0^+, we assume that it is linear and that

$$Ku(0) = 0.5K. \qquad (12.4)$$

$$Ku(t - a) = 0, \quad t < a,$$
$$Ku(t - a) = K, \quad t > a.$$



Laplace transform

- **Singularity functions**

Sometimes the derivative function of the derivative needs to be defined in order for the derivative of the function to have laplace transform. For that we use impulse function.

Mathematically, the **impulse function** is defined

$$\int_{-\infty}^{\infty} K\delta(t)dt = K;$$

$$\delta(t) = 0, \quad t \neq 0.$$

$$\mathcal{L}\{\delta^{(n)}(t)\} = s^n.$$

$$\delta(t) = \frac{du(t)}{dt}$$

Laplace transform

- **Laplace transform properties (functional and operational)**

Property	$f(t)$	$F(s)$
Linearity	$a_1 f_1(t) + a_2 f_2(t)$	$a_1 F_1(s) + a_2 F_2(s)$
Scaling	$f(at)$	$\frac{1}{a}F\left(\frac{s}{a}\right)$
Time shift	$f(t-a)u(t-a)$	$e^{-as}F(s)$
Frequency shift	$e^{-at}f(t)$	$F(s+a)$
Time differentiation	$\frac{df}{dt}$	$sF(s) - f(0^-)$
	$\frac{d^2f}{dt^2}$	$s^2F(s) - sf(0^-) - f'(0^-)$
	$\frac{d^3f}{dt^3}$	$s^3F(s) - s^2f(0^-) - sf'(0^-) - f''(0^-)$
	$\frac{d^nf}{dt^n}$	$s^nF(s) - s^{n-1}f(0^-) - s^{n-2}f'(0^-) - \cdots - f^{(n-1)}(0^-)$
Time integration	$\int_0^t f(x)\,dx$	$\frac{1}{s}F(s)$
Frequency differentiation	$tf(t)$	$-\frac{d}{ds}F(s)$
Frequency integration	$\frac{f(t)}{t}$	$\int_s^\infty F(s)\,ds$
Time periodicity	$f(t) = f(t+nT)$	$\frac{F_1(s)}{1-e^{-sT}}$
Initial value	$f(0)$	$\lim_{s\to\infty} sF(s)$
Final value	$f(\infty)$	$\lim_{s\to 0} sF(s)$
Convolution	$f_1(t) * f_2(t)$	$F_1(s)F_2(s)$

$f(t)$	$F(s)$
$\delta(t)$	1
$u(t)$	$\frac{1}{s}$
e^{-at}	$\frac{1}{s+a}$
t	$\frac{1}{s^2}$
t^n	$\frac{n!}{s^{n+1}}$
te^{-at}	$\frac{1}{(s+a)^2}$
t^ne^{-at}	$\frac{n!}{(s+a)^{n+1}}$
$\sin \omega t$	$\frac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\frac{s}{s^2+\omega^2}$
$\sin(\omega t + \theta)$	$\frac{s\sin\theta + \omega\cos\theta}{s^2+\omega^2}$
$\cos(\omega t + \theta)$	$\frac{s\cos\theta - \omega\sin\theta}{s^2+\omega^2}$
$e^{-at}\sin \omega t$	$\frac{\omega}{(s+a)^2+\omega^2}$
$e^{-at}\cos \omega t$	$\frac{s+a}{(s+a)^2+\omega^2}$

*Defined for $t \geq 0$; $f(t) = 0$, for $t < 0$.

Laplace transform

• Inverse Laplace transform: we avoid using the complex integral of the inverse laplace by using the operational and functional properties of the transforms.

• Having a specific laplace domain function, we can use partial fraction technique to find the inverse.

Suppose $F(s)$ has the general form of

$$F(s) = \frac{N(s)}{D(s)}$$

N(s): Numerator
D(s): Denominator

Poles are the roots of the denominator

Having a specific laplace domain function, we can use partial fraction technique to find the inverse. And then use the table to get back to the time domain.

Please review the four cases of partial fraction: simple poles, repeated poles complex poles and repeated complex poles

Laplace transform

• **Initial and final value theorems:** Before making the effort to perform the inverse transform of a response in the laplace domain, it is better to check whether it agrees with the expected behaviour of the system or not. And if it does agree, then we perform the transform. To check that we use the initial and final value theorems.

$$\lim_{t \to 0^+} f(t) = \lim_{s \to \infty} sF(s),$$ initial value theorem()

$$\lim_{t \to \infty} f(t) = \lim_{s \to 0} sF(s).$$ initial value theorem

Laplace transform

- **Steps for solving the circuit in laplace domain**

Steps in Applying the Laplace Transform:

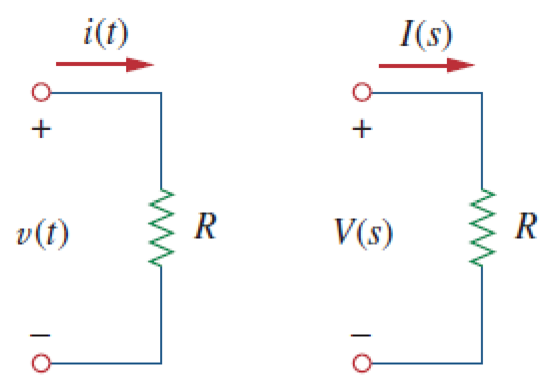
1. Transform the circuit from the time domain to the s-domain.
2. Solve the circuit using nodal analysis, mesh analysis, source transformation, superposition, or any circuit analysis technique with which we are familiar.
3. Take the inverse transform of the solution and thus obtain the solution in the time domain.

- **Component models in the laplace domain**

Resistor: $v(t) = Ri(t)$

$$\boxed{V(s) = RI(s)}$$

What is the unit of I(s) and V(s).



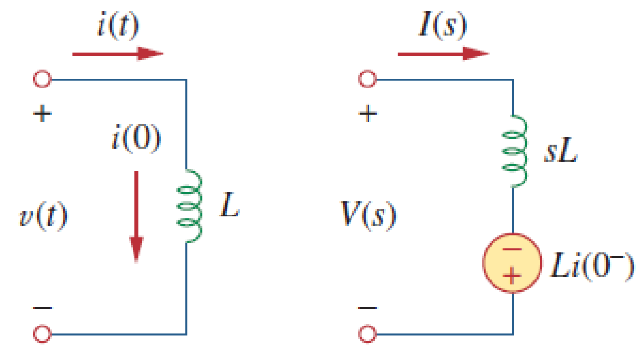
Laplace transform

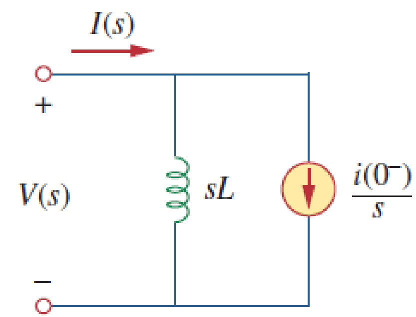
- **Components in the laplace domain**

Inductor: $v(t) = L\dfrac{di(t)}{dt}$

$$V(s) = L[sI(s) - i(0^-)] = sLI(s) - Li(0^-)$$

$$\boxed{I(s) = \frac{1}{sL}V(s) + \frac{i(0^-)}{s}}$$

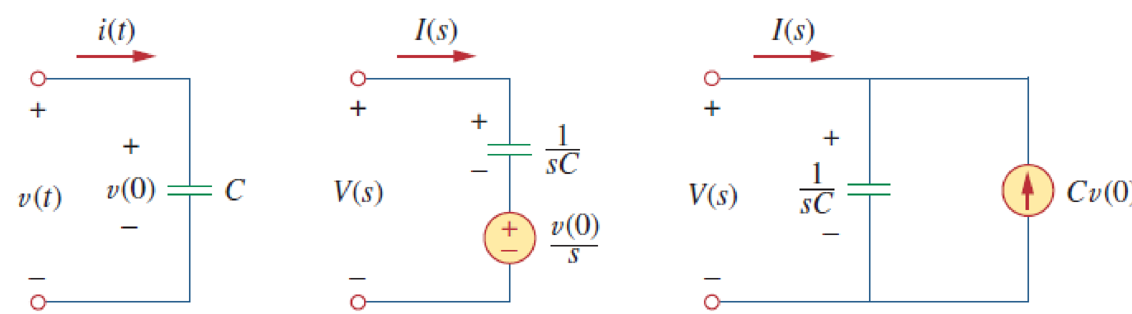




Capacitor: $i(t) = C\dfrac{dv(t)}{dt}$

$$I(s) = C[sV(s) - v(0^-)] = sCV(s) - Cv(0^-)$$

$$\boxed{V(s) = \frac{1}{sC}I(s) + \frac{v(0^-)}{s}}$$



Laplace transform

- **Impedances in the laplace domain: they are defined as:** $Z(s) = \frac{V(s)}{I(s)}$ **When the initial condition is zero**

Thus, the impedances of the three circuit elements are

$$\text{Resistor:} \quad Z(s) = R$$

$$\text{Inductor:} \quad Z(s) = sL$$

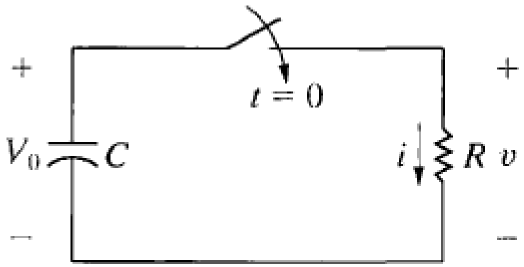
$$\text{Capacitor:} \quad Z(s) = \frac{1}{sC}$$

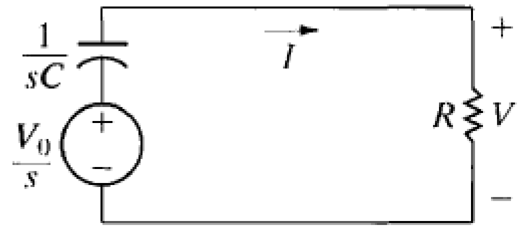
- **Addmittance:** $Y(s) = \frac{1}{Z(s)} = \frac{I(s)}{V(s)}$
- **Dependent sources** $\mathcal{L}[av(t)] = aV(s)$

$$\mathcal{L}[ai(t)] = aI(s)$$

Laplace transform

- **Natural response of RC circuit (in time domain and laplace domain):**





$$C\frac{dV}{dt}+\frac{V}{R}=0 \Rightarrow \frac{dV}{dt}+\frac{V}{RC}=0$$

$$\Rightarrow a=\frac{1}{RC},\ \mathrm{V}(\infty)=0,\ \mathrm{V}(0)=V_0$$

$$V(t)=V(\infty)+(V(0)-V(\infty))e^{(-at)}=V_0e^{(-t/RC)}$$

$$V_C(t)=V_0e^{(-t/RC)}$$

$$\tau=R_{equ}C_{equ}$$

Test using initial and final value theorems!!

$$\frac{V_0}{s}=\frac{1}{sC}I+RI.$$

$$I=\frac{CV_0}{RCs+1}=\frac{V_0/R}{s+(1/RC)}$$

$$i=\frac{V_0}{R}e^{-t/RC}u(t),$$

$$v=Ri=V_0e^{-t/RC}u(t)$$

Laplace transform

- **Example**

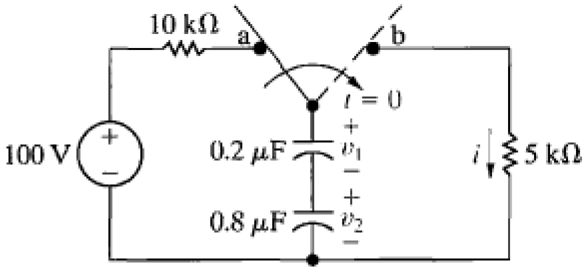
13.3 The switch in the circuit shown has been in position a for a long time. At $t = 0$, the switch is thrown to position b.

a) Find I, V_1, and V_2 as rational functions of s.

b) Find the time-domain expressions for i, v_1, and v_2.

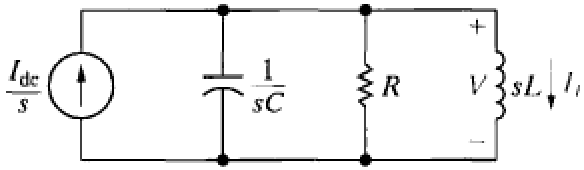
Answer: (a) $I = 0.02/(s + 1250)$,
$V_1 = 80/(s + 1250)$,
$V_2 = 20/(s + 1250)$;

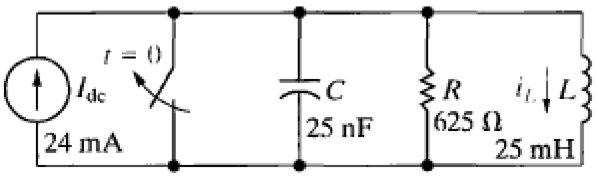
(b) $i = 20e^{-1250t}u(t)$ mA,
$v_1 = 80e^{-1250t}u(t)$ V,
$v_2 = 20e^{-1250t}u(t)$ V.



Laplace transform

- **Step response of parallel RLC**





Nodal analysis:

$$sCV + \frac{V}{R} + \frac{V}{sL} = \frac{I_{dc}}{s}.$$

$$V = \frac{I_{dc}/C}{s^2 + (1/RC)s + (1/LC)}.$$

$$I_L = \frac{I_{dc}/LC}{s[s^2 + (1/RC)s + (1/LC)]}.$$

$$I_L = \frac{384 \times 10^5}{s(s^2 + 64{,}000s + 16 \times 10^8)}.$$

$$I_L = \frac{384 \times 10^5}{s(s + 32{,}000 - j24{,}000)(s + 32{,}000 + j24{,}000)}.$$

$$\lim_{s \to 0} sI_L = \frac{384 \times 10^5}{16 \times 10^8} = 24 \text{ mA}.$$

$$I_L = \frac{K_1}{s} + \frac{K_2}{s + 32{,}000 - j24{,}000} + \frac{K_2^*}{s + 32{,}000 + j24{,}000}.$$

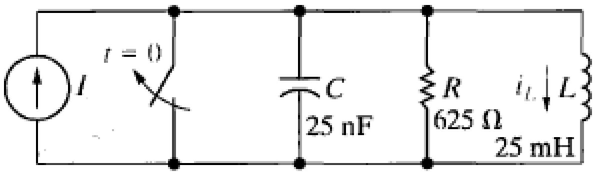
$$K_1 = \frac{384 \times 10^5}{16 \times 10^8} = 24 \times 10^{-3},$$

$$K_2 = \frac{384 \times 10^5}{(-32{,}000 + j24{,}000)(j48{,}000)} = 20 \times 10^{-3} \underline{/126.87^\circ}.$$

$$i_L = [24 + 40e^{-32{,}000t} \cos(24{,}000t + 126.87^\circ)]u(t) \text{mA}.$$

Laplace transform

- **Transient response of parallel RLC sinusoidal input**



$$i_g = I_m \cos \omega t \text{ A}, \tag{13.30}$$

where $I_m = 24$ mA and $\omega = 40{,}000$ rad/s. As before, we assume that the initial energy stored in the circuit is zero.

The s-domain expression for the source current is

$$I_g = \frac{sI_m}{s^2 + \omega^2}. \tag{13.31}$$

The voltage across the parallel elements is

$$V = \frac{(I_g/C)s}{s^2 + (1/RC)s + (1/LC)}. \tag{13.32}$$

Substituting Eq. 13.31 into Eq. 13.32 results in

$$V = \frac{(I_m/C)s^2}{(s^2 + \omega^2)[s^2 + (1/RC)s + (1/LC)]}, \tag{13.33}$$

from which

$$I_L = \frac{V}{sL} = \frac{(I_m/LC)s}{(s^2 + \omega^2)[s^2 + (1/RC)s + (1/LC)]}. \tag{13.34}$$

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Laplace transform

- **Transient response of parallel RLC sinusoidal input**

Substituting the numerical values of I_m, ω, R, L, and C into Eq. 13.34 gives

$$I_L = \frac{384 \times 10^5 s}{(s^2 + 16 \times 10^8)(s^2 + 64{,}000s + 16 \times 10^8)}. \qquad (13.35)$$

We now write the denominator in factored form:

$$I_L = \frac{384 \times 10^5 s}{(s - j\omega)(s + j\omega)(s + \alpha - j\beta)(s + \alpha + j\beta)}, \qquad (13.36)$$

$$I_L = \frac{K_1}{s - j40{,}000} + \frac{K_1^*}{s + j40{,}000} + \frac{K_2}{s + 32{,}000 - j24{,}000}$$

$$+ \frac{K_2^*}{s + 32{,}000 + j24{,}000}. \qquad (13.37)$$

The numerical values of the coefficients K_1 and K_2 are

$$K_1 = \frac{384 \times 10^5 (j40{,}000)}{(j80{,}000)(32{,}000 + j16{,}000)(32{,}000 + j64{,}000)}$$

$$= 7.5 \times 10^{-3} \underline{/-90^\circ}, \qquad (13.38)$$

Laplace transform

- **Transient response of parallel RLC sinusoidal input**

$$K_2 = \frac{384 \times 10^5(-32{,}000 + j24{,}000)}{(-32{,}000 - j16{,}000)(-32{,}000 + j64{,}000)(j48{,}000)}$$

$$= 12.5 \times 10^{-3} \angle 90^\circ. \qquad (13.39)$$

Substituting the numerical values from Eqs. 13.38 and 13.39 into Eq. 13.37 and inverse-transforming the resulting expression yields

$$i_L = [15 \cos(40{,}000t - 90^\circ)$$

$$+ 25e^{-32{,}000t} \cos(24{,}000t + 90^\circ)] \text{ mA},$$

$$= (15 \sin 40{,}000t - 25e^{-32{,}000t} \sin 24{,}000t)u(t) \text{ mA}. \qquad (13.40)$$

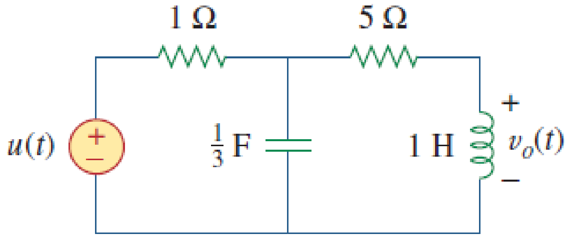
We now test Eq. 13.40 to see whether it makes sense in terms of the given initial conditions and the known circuit behavior after the switch has been open for a long time. For $t = 0$, Eq. 13.40 predicts zero initial current, which agrees with the initial energy of zero in the circuit. Equation 13.40 also predicts a steady-state current of

$$i_{L_{ss}} = 15 \sin 40{,}000t \text{ mA}, \qquad (13.41)$$

Laplace transform

- **Example 1:**

Find $v_o(t)$ in the circuit of Fig. 16.4, assuming zero initial conditions.

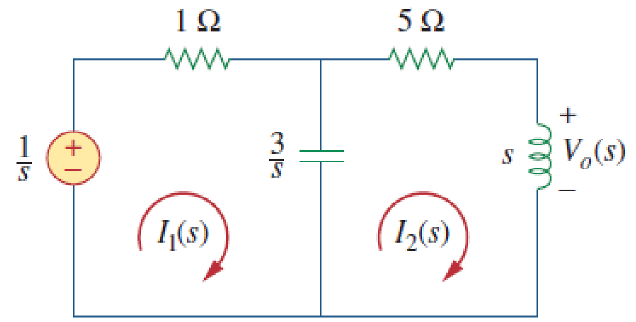


Solution:
We first transform the circuit from the time domain to the s-domain.

$$u(t) \quad \Rightarrow \quad \frac{1}{s}$$

$$1\ \text{H} \quad \Rightarrow \quad sL = s$$

$$\frac{1}{3}\text{F} \quad \Rightarrow \quad \frac{1}{sC} = \frac{3}{s}$$



The resulting s-domain circuit is in Fig. 16.5. We now apply mesh analysis. For mesh 1,

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)I_1 - \frac{3}{s}I_2 \qquad \textbf{(16.1.1)}$$

For mesh 2,

$$0 = -\frac{3}{s}I_1 + \left(s + 5 + \frac{3}{s}\right)I_2$$

Laplace transform

- **Example 1:**

or

$$I_1 = \frac{1}{3}(s^2 + 5s + 3)I_2 \qquad \textbf{(16.1.2)}$$

Substituting this into Eq. (16.1.1),

$$\frac{1}{s} = \left(1 + \frac{3}{s}\right)\frac{1}{3}(s^2 + 5s + 3)I_2 - \frac{3}{s}I_2$$

Multiplying through by $3s$ gives

$$3 = (s^3 + 8s^2 + 18s)I_2 \qquad \Rightarrow \qquad I_2 = \frac{3}{s^3 + 8s^2 + 18s}$$

$$V_o(s) = sI_2 = \frac{3}{s^2 + 8s + 18} = \frac{3}{\sqrt{2}}\frac{\sqrt{2}}{(s + 4)^2 + (\sqrt{2})^2}$$

Taking the inverse transform yields

$$v_o(t) = \frac{3}{\sqrt{2}}e^{-4t}\sin\sqrt{2}t\ \text{V}, \qquad t \geq 0$$

Laplace transform

- **Example 2:**

Find $v_o(t)$ in the circuit of Fig. 16.7. Assume $v_o(0) = 5$ V.

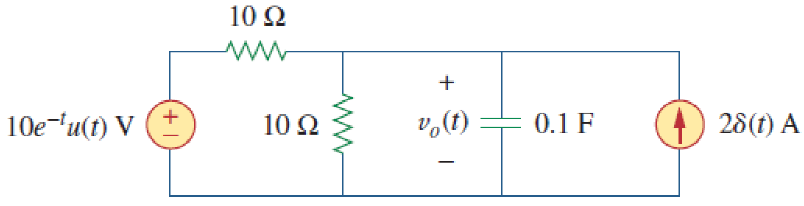


Figure 16.7
For Example 16.2.

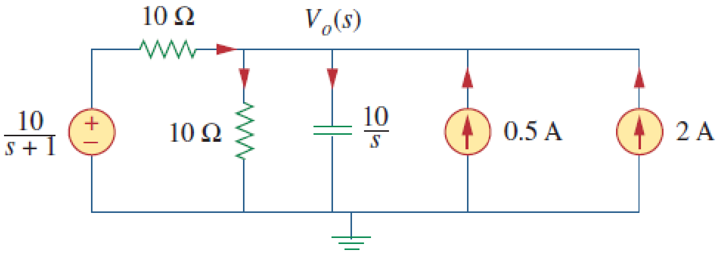


Figure 16.8
Nodal analysis of the equivalent of the circuit in Fig. 16.7.

Solution:

We transform the circuit to the s-domain as shown in Fig. 16.8. The initial condition is included in the form of the current source $Cv_o(0) = 0.1(5) = 0.5$ A. [See Fig. 16.2(c).] We apply nodal analysis. At the top node,

$$\frac{10/(s+1) - V_o}{10} + 2 + 0.5 = \frac{V_o}{10} + \frac{V_o}{10/s}$$

or

$$\frac{1}{s+1} + 2.5 = \frac{2V_o}{10} + \frac{sV_o}{10} = \frac{1}{10}V_o(s+2)$$

Laplace transform

- **Example 2:**

Multiplying through by 10,

$$\frac{10}{s+1} + 25 = V_o(s+2)$$

or

$$V_o = \frac{25s+35}{(s+1)(s+2)} = \frac{A}{s+1} + \frac{B}{s+2}$$

where

$$A = (s+1)V_o(s)\,\big|_{s=-1} = \frac{25s+35}{(s+2)}\bigg|_{s=-1} = \frac{10}{1} = 10$$

$$B = (s+2)V_o(s)\,\big|_{s=-2} = \frac{25s+35}{(s+1)}\bigg|_{s=-2} = \frac{-15}{-1} = 15$$

Thus,

$$V_o(s) = \frac{10}{s+1} + \frac{15}{s+2}$$

Taking the inverse Laplace transform, we obtain

$$v_o(t) = (10e^{-t} + 15e^{-2t})u(t)\ \text{V}$$

Laplace transform

- **Example 3:**

In the circuit of Fig. 16.10(a), the switch moves from position a to position b at $t = 0$. Find $i(t)$ for $t > 0$.

Solution:

The initial current through the inductor is $i(0) = I_o$. For $t > 0$, Fig. 16.10(b) shows the circuit transformed to the s-domain. The initial condition is incorporated in the form of a voltage source as $Li(0) = LI_o$. Using mesh analysis,

$$I(s)(R + sL) - LI_o - \frac{V_o}{s} = 0 \qquad \textbf{(16.3.1)}$$

or

$$I(s) = \frac{LI_o}{R + sL} + \frac{V_o}{s(R + sL)} = \frac{I_o}{s + R/L} + \frac{V_o/L}{s(s + R/L)} \qquad \textbf{(16.3.2)}$$

Applying partial fraction expansion on the second term on the right-hand side of Eq. (16.3.2) yields

$$I(s) = \frac{I_o}{s + R/L} + \frac{V_o/R}{s} - \frac{V_o/R}{(s + R/L)} \qquad \textbf{(16.3.3)}$$

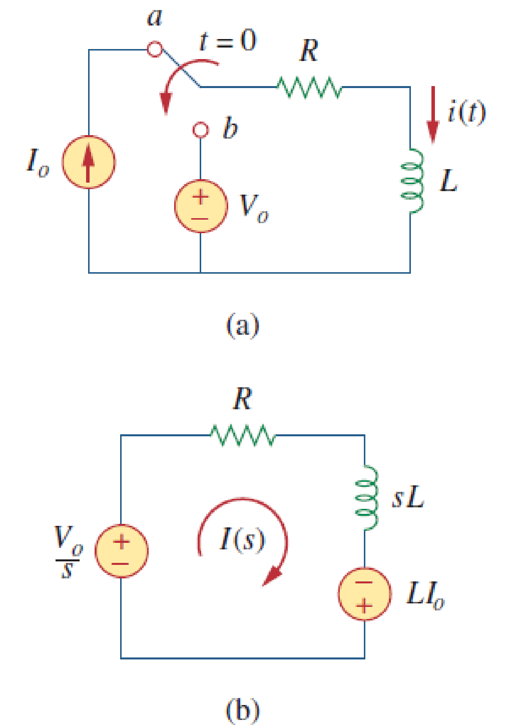


Figure 16.10

Laplace transform

- **Example 3:**

The inverse Laplace transform of this gives

$$i(t) = \left(I_o - \frac{V_o}{R}\right)e^{-t/\tau} + \frac{V_o}{R}, \qquad t \geq 0 \qquad \textbf{(16.3.4)}$$

where $\tau = R/L$. The term in parentheses is the transient response, while the second term is the steady-state response. In other words, the final value is $i(\infty) = V_o/R$, which we could have predicted by applying the final-value theorem on Eq. (16.3.2) or (16.3.3); that is,

$$\lim_{s\to 0} sI(s) = \lim_{s\to 0}\left(\frac{sI_o}{s + R/L} + \frac{V_o/L}{s + R/L}\right) = \frac{V_o}{R} \qquad \textbf{(16.3.5)}$$

Equation (16.3.4) may also be written as

$$i(t) = I_o e^{-t/\tau} + \frac{V_o}{R}(1 - e^{-t/\tau}), \qquad t \geq 0 \qquad \textbf{(16.3.6)}$$

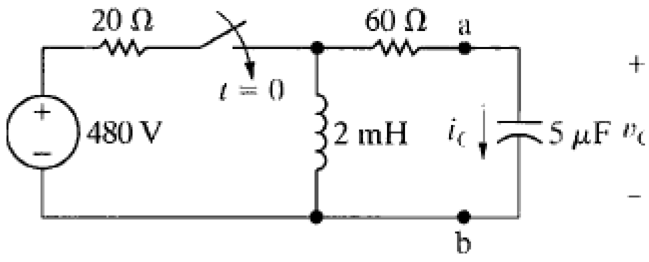
The first term is the natural response, while the second term is the forced response. If the initial condition $I_o = 0$, Eq. (16.3.6) becomes

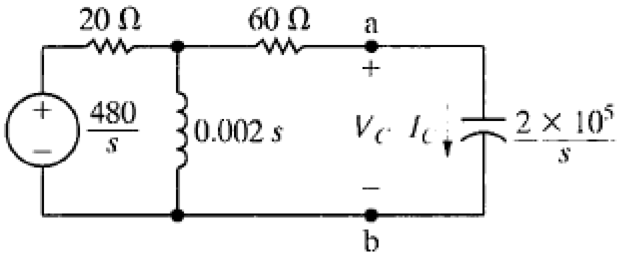
$$i(t) = \frac{V_o}{R}(1 - e^{-t/\tau}), \qquad t \geq 0 \qquad \textbf{(16.3.7)}$$

which is the step response, since it is due to the step input V_o with no initial energy.

Laplace transform

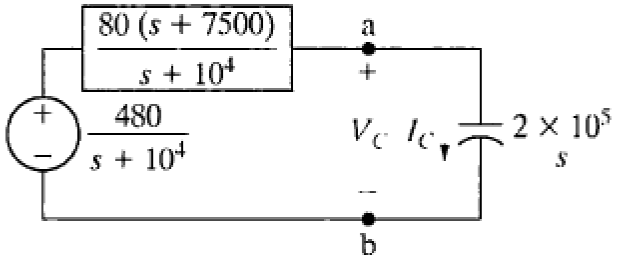
- **Thevenin's equivalent circuit:**





$$V_{Th} = \frac{(480/s)(0.002s)}{20 + 0.002s} = \frac{480}{s + 10^4}.$$

$$Z_{Th} = 60 + \frac{0.002s(20)}{20 + 0.002s} = \frac{80(s + 7500)}{s + 10^4}.$$



$$I_C = \frac{480/(s + 10^4)}{[80(s + 7500)/(s + 10^4)] + [(2 \times 10^5)/s]}.$$

$$I_C = \frac{6s}{s^2 + 10{,}000s + 25 \times 10^6} = \frac{6s}{(s + 5000)^2}.$$

$$I_C = \frac{-30{,}000}{(s + 5000)^2} + \frac{6}{s + 5000},$$

$$i_C = (-30{,}000te^{-5000t} + 6e^{-5000t})u(t)\text{ A}.$$

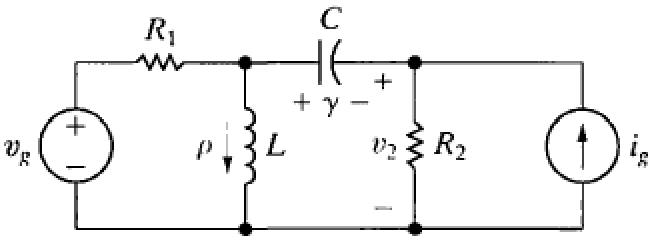
$$V_C = \frac{1}{sC}I_C = \frac{2 \times 10^5}{s}\,\frac{6s}{(s + 5000)^2}$$

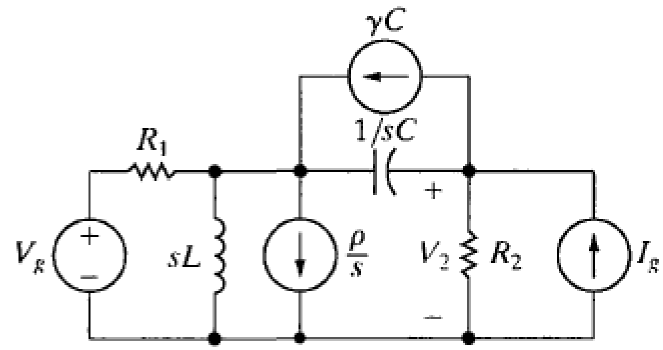
$$= \frac{12 \times 10^5}{(s + 5000)^2},$$

$$v_C = 12 \times 10^5 te^{-5000t}u(t).$$

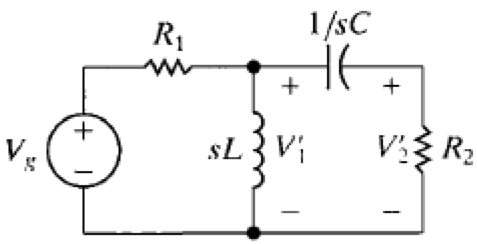
Laplace transform

- **Superposition theory**

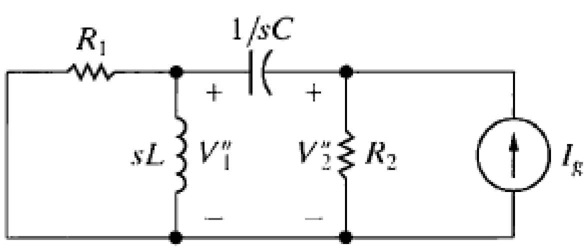




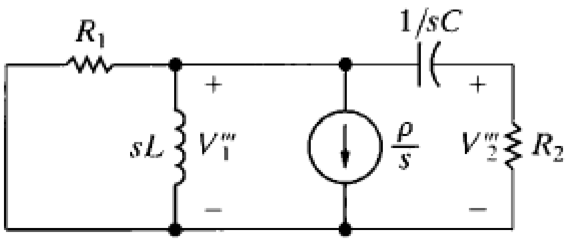
Source 1:



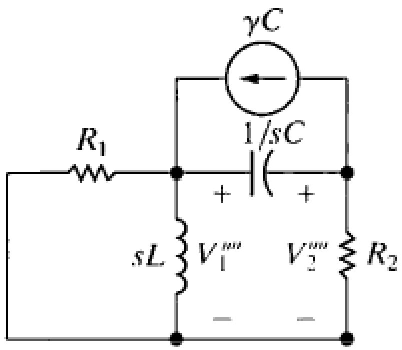
Source 2:



Source 3:



Source 4:



$$V_2 = V_2' + V_2'' + V_2''' + V_2''''$$

Laplace transform

- **Network Functions (Transfer functions and input impedance)**

The *transfer function* is a key concept in signal processing because it indicates how a signal is processed as it passes through a network. It is a fitting tool for finding the network response, determining (or designing for) network stability, and network synthesis. The transfer function of a network describes how the output behaves with respect to the input. It specifies the transfer from the input to the output in the s-domain, assuming no initial energy.

The transfer function depends on what we define as input and output. Since the input and output can be either current or voltage at any place in the circuit, there are four possible transfer functions:

$$T_V(s) = \text{Voltage Transfer Function} = \frac{V_2(s)}{V_1(s)}$$

$$T_I(s) = \text{Current Transfer Function} = \frac{I_2(s)}{I_1(s)}$$

$$T_Y(s) = \text{Transfer Admittance} = \frac{I_2(s)}{V_1(s)}$$

$$T_Z(s) = \text{Transfer Impedance} = \frac{V_2(s)}{I_1(s)}$$

Driving point impedance or admittance (voltage to current on the same port.

$$H(s) = \text{Impedance} = \frac{V(s)}{I(s)}$$

$$H(s) = \text{Admittance} = \frac{I(s)}{V(s)}$$

The **transfer function** $H(s)$ is the ratio of the output response $Y(s)$ to the input excitation $X(s)$, assuming all initial conditions are zero.

$$\boxed{H(s) = \frac{Y(s)}{X(s)}}$$

Laplace transform

- **The Transfer Function Example:**

Example 13.1 **Deriving the Transfer Function of a Circ**

The voltage source v_g drives the circuit shown in Fig. 13.31. The response signal is the voltage across the capacitor, v_o.

a) Calculate the numerical expression for the transfer function.

b) Calculate the numerical values for the poles and zeros of the transfer function.

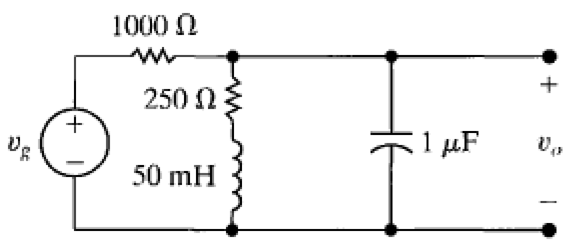


Figure 13.31 ▲ The circuit for Example 13.1.

Solution

a) The first step in finding the transfer function is to construct the s-domain equivalent circuit, as shown in Fig. 13.32. By definition, the transfer function is the ratio of V_o/V_g, which can be computed from a single node-voltage equation. Summing the currents away from the upper node generates

$$\frac{V_o - V_g}{1000} + \frac{V_o}{250 + 0.05s} + \frac{V_o s}{10^6} = 0.$$

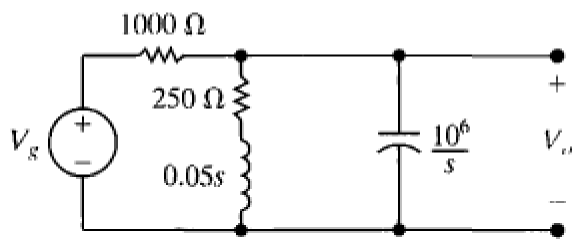


Figure 13.32 ▲ The s-domain equivalent circuit for the circuit shown in Fig. 13.31.

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Laplace transform

- **The Transfer Function Example:**

Solving for V_o yields

$$V_o = \frac{1000(s + 5000)V_g}{s^2 + 6000s + 25 \times 10^6}.$$

Hence the transfer function is

$$H(s) = \frac{V_o}{V_g}$$

$$= \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}.$$

b) The poles of $H(s)$ are the roots of the denominator polynomial. Therefore

$$-p_1 = -3000 - j4000,$$

$$-p_2 = -3000 + j4000.$$

The zeros of $H(s)$ are the roots of the numerator polynomial; thus $H(s)$ has a zero at

$$-z_1 = -5000.$$

Important

Thus, because circuits may have multiple sources and because the definition of the output signal of interest can vary, a single circuit can generate many transfer functions. Remember that when multiple sources are involved, no single transfer function can represent the total output—transfer functions associated with each source must be combined using superposition to yield the total response.

Laplace transform

- **The transfer function in partial fraction: The transfer function is a rational function and can be expanded in partial fraction, the poles resulting from the transfer function are responsible for the transient effect in the response. But the input poles are responsible for the steady state.**

Can there be transient response without initial conditions?
Can there be natural response without initial conditions?
Pole zero diagrams
Comment on impulse responce and time invariance.

Laplace transform

The circuit in Example 13.1 (Fig. 13.31) is driven by a voltage source whose voltage increases linearly with time, namely, $v_g = 50tu(t)$.

a) Use the transfer function to find v_o.

b) Identify the transient component of the response.

c) Identify the steady-state component of the response.

d) Sketch v_o versus t for $0 \le t \le 1.5$ ms.

Solution

a) From Example 13.1,

$$H(s) = \frac{1000(s + 5000)}{s^2 + 6000s + 25 \times 10^6}.$$

The transform of the driving voltage is $50/s^2$; therefore, the s-domain expression for the output voltage is

$$V_o = \frac{1000(s + 5000)}{(s^2 + 6000s + 25 \times 10^6)} \frac{50}{s^2}.$$

The partial fraction expansion of V_o is

$$V_o = \frac{K_1}{s + 3000 - j4000} + \frac{K_1^*}{s + 3000 + j4000} + \frac{K_2}{s^2} + \frac{K_3}{s}.$$

We evaluate the coefficients K_1, K_2, and K_3 by using the techniques described in Section 12.7:

$$K_1 = 5\sqrt{5} \times 10^{-4} \angle 79.70^\circ;$$

$$K_1^* = 5\sqrt{5} \times 10^{-4} \angle -79.70^\circ,$$

$$K_2 = 10,$$

$$K_3 = -4 \times 10^{-4}.$$

The time-domain expression for v_o is

$$v_o = [10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.70^\circ) + 10t - 4 \times 10^{-4}]u(t) \text{ V}.$$

Laplace transform

b) The transient component of v_o is

$$10\sqrt{5} \times 10^{-4} e^{-3000t} \cos(4000t + 79.70°).$$

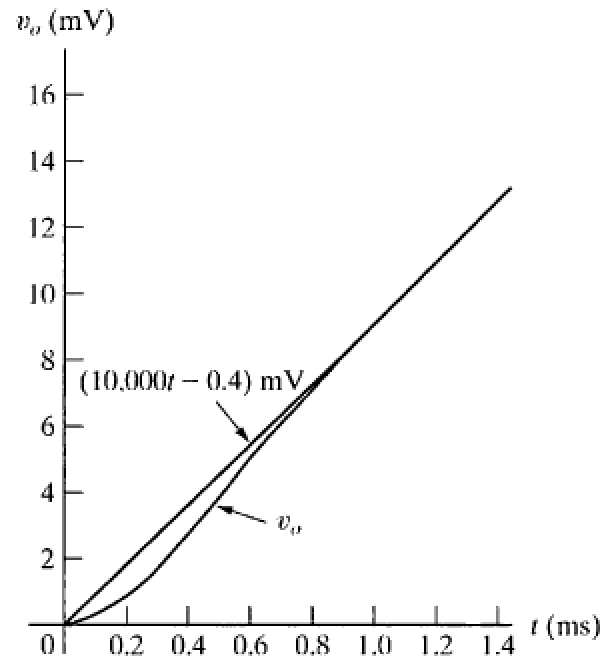
Note that this term is generated by the poles $(-3000 + j4000)$ and $(-3000 - j4000)$ of the transfer function.

c) The steady-state component of the response is

$$(10t - 4 \times 10^{-4})u(t).$$

These two terms are generated by the second-order pole (K/s^2) of the driving voltage.

d) Figure 13.33 shows a sketch of v_o versus t. Note that the deviation from the steady-state solution $10{,}000t - 0.4$ mV is imperceptible after approximately 1 ms.



Laplace transform

The transfer function and the steady state response

Once we have computed a circuit's transfer function, we no longer need to perform a separate phasor analysis of the circuit to determine its steady-state response. Instead, we use the transfer function to relate the steady-state response to the excitation source. First we assume that

$$x(t) = A\cos(\omega t + \phi),$$

$$x(t) = A\cos\omega t\cos\phi - A\sin\omega t\sin\phi,$$

$$X(s) = \frac{(A\cos\phi)s}{s^2+\omega^2} - \frac{(A\sin\phi)\omega}{s^2+\omega^2}$$

$$= \frac{A(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2}.$$

$$Y(s) = H(s)\frac{A(s\cos\phi - \omega\sin\phi)}{s^2+\omega^2}$$

$$Y(s) = \frac{K_1}{s-j\omega} + \frac{K_1^*}{s+j\omega}$$

$$+ \sum \text{terms generated by the poles of } H(s).$$

$$K_1 = \left.\frac{H(s)A(s\cos\phi - \omega\sin\phi)}{s+j\omega}\right|_{s=j\omega}$$

$$= \frac{H(j\omega)A(j\omega\cos\phi - \omega\sin\phi)}{2j\omega}$$

$$= \frac{H(j\omega)A(\cos\phi + j\sin\phi)}{2} = \frac{1}{2}H(j\omega)Ae^{j\phi}.$$

$$H(j\omega) = |H(j\omega)|e^{j\theta(\omega)}$$

$$K_1 = \frac{A}{2}|H(j\omega)|e^{j[\theta(\omega)+\phi]}$$

$$y_{ss}(t) = A|H(j\omega)|\cos[\omega t + \phi + \theta(\omega)]$$

Laplace transform

The transfer function and the steady state response

Example 13.4 **Using the Transfer Function to Find the Steady-State Sinusoidal Response**

The circuit from Example 13.1 is shown in Fig. 13.46. The sinusoidal source voltage is $120 \cos(5000t + 30^\circ)$ V. Find the steady-state expression for v_o.

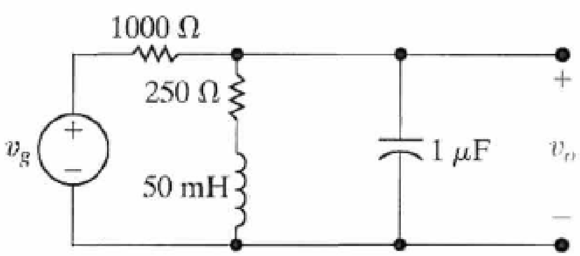


Figure 13.46 ▲ The circuit for Example 13.4.

Solution

From Example 13.1,

$$H(s) = \frac{1000(s + 5000)}{s^2 + 6000s + 25 * 10^6}.$$

The frequency of the voltage source is 5000 rad/s; hence we evaluate $H(s)$ at $H(j5000)$:

$$H(j5000) = \frac{1000(5000 + j5000)}{-25 * 10^6 + j5000(6000) + 25 \times 10^6}$$

$$= \frac{1 + j1}{j6} = \frac{1 - j1}{6} = \frac{\sqrt{2}}{6}\angle -45^\circ.$$

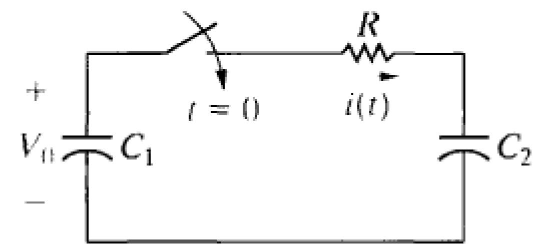
Then, from Eq. 13.120,

$$v_{o_{ss}} = \frac{(120)\sqrt{2}}{6} \cos(5000t + 30^\circ - 45^\circ)$$

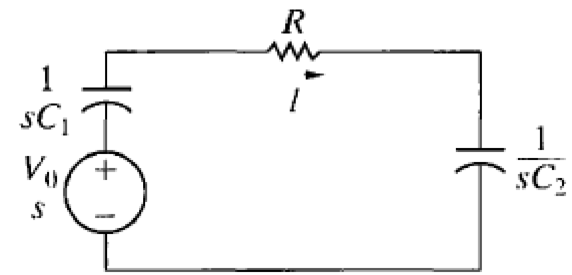
$$= 20\sqrt{2} \cos(5000t - 15^\circ) \text{ V}.$$

Laplace transform

How to generate impulses:



Impulse functions occur in circuit analysis either because of a switching operation or because a circuit is excited by an impulsive source. The Laplace transform can be used to predict the impulsive currents and voltages created during switching and the response of a circuit to an impulsive source. We begin our discussion by showing how to create an impulse function with a switching operation.



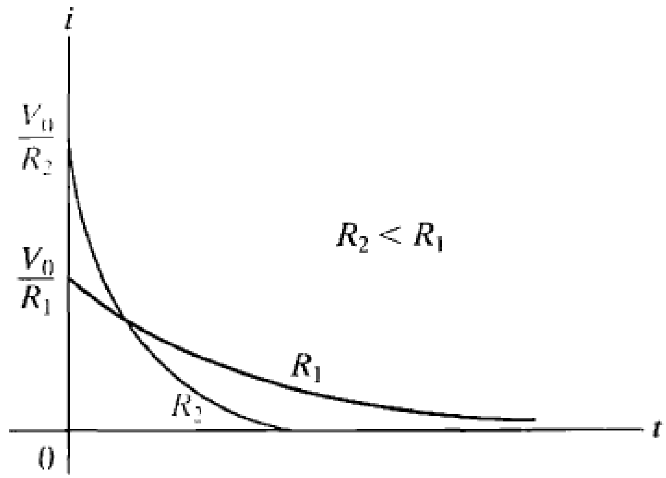
$$I = \frac{V_0/s}{R + (1/sC_1) + (1/sC_2)}$$

$$= \frac{V_0/R}{s + (1/RC_e)},$$

$$i = \left(\frac{V_0}{R}e^{-t/RC_e}\right)u(t)$$

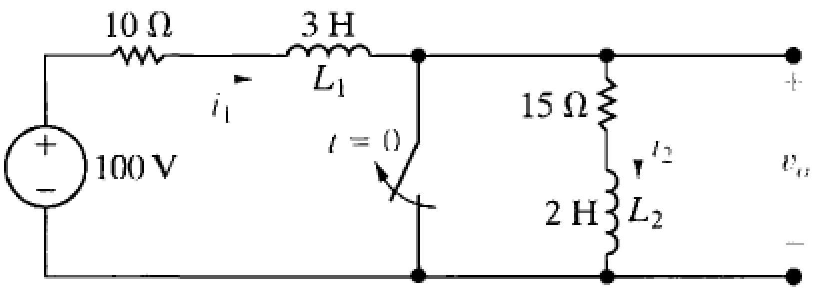
$$\text{Area} = q = \int_{0^-}^{\infty}\frac{V_0}{R}e^{-t/RC_e}dt = V_0C_e, \quad i \to V_0C_e\delta(t).$$

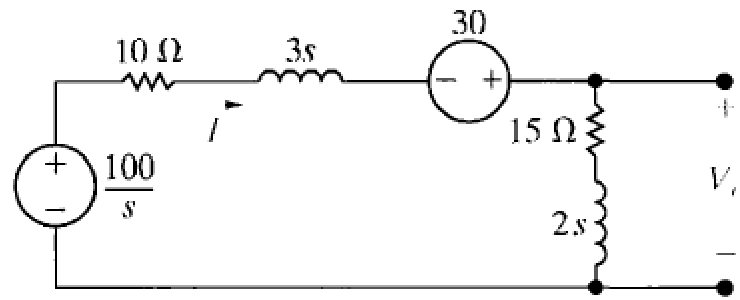
$$I = \frac{V_0/s}{(1/sC_1) + (1/sC_2)} = \frac{C_1C_2V_0}{C_1 + C_2} = C_eV_0.$$



Laplace transform

How to generate impulses:





$$\frac{V_o}{2s + 15} + \frac{V_o - [(100/s) + 30]}{3s + 10} = 0.$$

$$V_o = \frac{40(s + 7.5)}{s(s + 5)} + \frac{12(s + 7.5)}{s + 5}.$$

$$V_o = \frac{60}{s} - \frac{20}{s + 5} + 12 + \frac{30}{s + 5}$$

$$= 12 + \frac{60}{s} + \frac{10}{s + 5},$$

$$v_o = 12\delta(t) + (60 + 10e^{-5t})u(t)\ \text{V}.$$

The ability of the Laplace transform to predict correctly the occurrence of an impulsive response is one reason why the transform is widely used to analyze the transient behavior of linear lumped-parameter time-invariant circuits.

Laplace transform

Benefits of impulse and impulse response:

Any signal can be represented by an infinite number of impulses given by

$$x(t) = \int_{-\infty}^{\infty} x(s)\,\delta(t - s)\,ds.$$

Knowing how the system respond to one impulse can lead to knowing the response to any signal, given that the system is linear and time invariant. This result is given by the convolution integral.

$$y(t) = \int_{-\infty}^{\infty} x(\lambda)h(t - \lambda)\,d\lambda,$$

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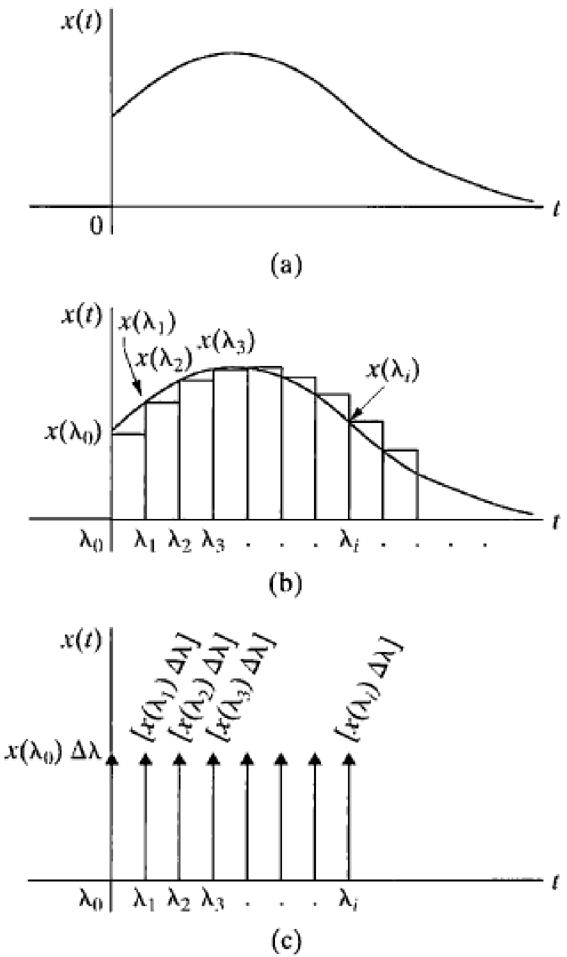
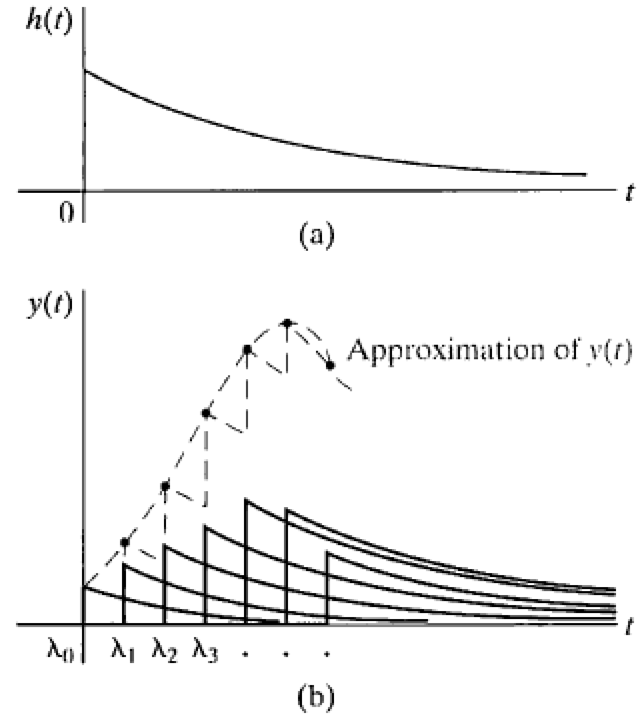
Laplace transform

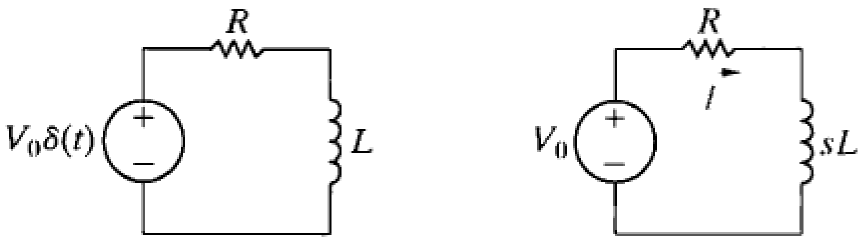
Impulse sources:





Laplace transform

Impulse sources: as if you have initial condition on the inductor released



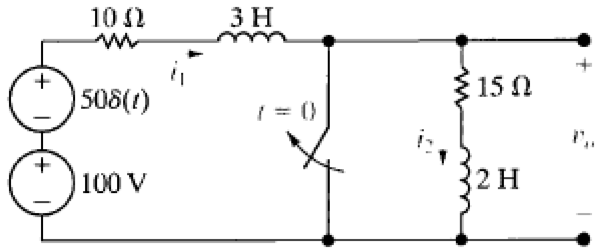
$$I = \frac{V_0}{R + sL} = \frac{V_0/L}{s + (R/L)},$$

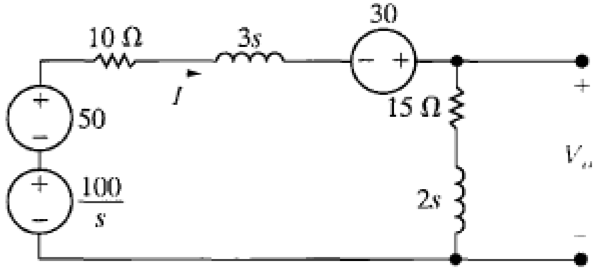
$$i = \frac{V_0}{L} e^{-(R/L)t} = \frac{V_0}{L} e^{-t/\tau} u(t).$$

Laplace transform

Impulse sources:

Finally, we consider the case in which internally generated impulses and externally applied impulses occur simultaneously. The Laplace transform approach automatically ensures the correct solution for $t > 0^+$ if inductor currents and capacitor voltages at $t = 0^-$ are used in constructing the s-domain equivalent circuit and if externally applied impulses are represented by their transforms. To illustrate, we add an impulsive voltage source of $50\delta(t)$ in series with the 100 V source to the circuit





$$I = \frac{50 + (100/s) + 30}{25 + 5s}$$

$$= \frac{16}{s + 5} + \frac{20}{s(s + 5)}$$

$$= \frac{16}{s + 5} + \frac{4}{s} - \frac{4}{s + 5}$$

$$= \frac{12}{s + 5} + \frac{4}{s}.$$

$$i(t) = (12e^{-5t} + 4)u(t) \text{ A}.$$

The expression for V_o is

$$V_o = (15 + 2s)I = \frac{32(s + 7.5)}{s + 5} + \frac{40(s + 7.5)}{s(s + 5)}$$

$$= 32\left(1 + \frac{2.5}{s + 5}\right) + \frac{60}{s} - \frac{20}{s + 5}$$

$$= 32 + \frac{60}{s + 5} + \frac{60}{s},$$

from which

$$v_o = 32\delta(t) + (60e^{-5t} + 60)u(t) \text{ V}.$$